A TEXT-BOOK

ON

ADVANCED ALGEBRA

AND

TRIGONOMETRY

WITH TABLES

BY

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NEW YORK
THE CENTURY CO.
1910

Published, August, 1910

Stanhope Press
F. H. GILSON COMPANY
BOSTON. U.S.A.

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PREFACE

In a considerable number of our colleges and universities the work of the first semester in mathematics is devoted to Algebra and Trigonometry. Usually Algebra is taken up first and then Trigonometry, or else the two subjects are studied on alternate days. Neither plan is quite satisfactory. It has therefore seemed to the writer that a single book, treating both subjects in a correlated manner, might be of service both to student and teacher.

In the present text the principal departures from the subject matter usually treated will be found in chapters 13 and 14. The chief aim has been to follow a mode and sequence of presentation which shall introduce the student who needs to apply his knowledge of mathematics in his other work as directly as possible to those facts and concepts which are most useful to him.

For this reason much stress is laid on graphic methods in the chapters on linear and quadratic equations, and this is followed up later as opportunity arises. It is thought that the extra time so used will be more than made up when the student begins his study of Analytical Geometry, because he will have become gradually familiar with the fundamental idea of this subject and need not readjust himself after an abrupt transition to a strange and mysterious realm.

For a similar reason the basic idea of the Differential Calculus is presented in a study of the derivative, and application is made to some of the simple standard functions. Maclaurin's formula is also obtained, and used to derive several standard expansions, among them the binomial theorem for any exponent.

A considerable emphasis has been placed on numerical computation, that the student may have some training in ready calculations. This can be largely supplemented by requiring students to work out mentally in class many of the numerical exercises.

It has been thought advisable to include some matter which may be omitted if only one semester is to be given to this course. Just what is to be omitted must of course be left to the judgment of the instructor.

W. C. B.

Lincoln, *March*, 1910.

ADVANCED ALGEBRA

AND

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ADVANCED ALGEBRA AND TRIGONOMETRY

CHAPTER I

THE OPERATIONS OF ALGEBRA

1. Letters as Symbols of Quantity. — In algebra, the letters of the alphabet are used to designate quantity or magnitude. Thus we speak of a line whose length is l feet, of a weight of w pounds, or of a velocity of v feet per second. Here the letter used, l, w, v, is suggested by the quantity considered, length, weight, velocity. When a number of different lines are considered, say n lines, their several lengths may be indicated by $l_1, l_2, l_3, \ldots, l_n$, or by $l^{(1)}$, $l^{(2)}, l^{(3)}, \ldots, l^{(n)}$. Three or four different lengths may be indicated by accents (called "primes"), as $l', l'', l''', \ldots$.

Fixed or known quantities are usually designated by the first letters of the alphabet, as by $a, b, c, \ldots$; unknown quantities which are to be determined from given data are represented by the last letters of the alphabet, as by $x, y, z, \ldots$. If x denote a quantity of a certain kind, other quantities of the same kind are indicated by $x_1, x_2, x_3, \ldots$ (read, "x sub-one, x sub-two, x sub-three, etc."), or by $x^{(1)}, x^{(2)}, x^{(3)}, \ldots$ (read "x superscript one, x superscript two, x superscript three, etc."), or by $x', x'', x''', \ldots$ (read "x prime, x second, x third, etc.").

2. Signs of Relation. — These are

$=$, read "equals," "is equal to," etc.;

$\neq$, read "is not equal to";

$<$, read "is less than";

$>$, read "is greater than";

$\nless$, read "is not less than";

$\ngtr$, read "is not greater than";

$\equiv$, read "is identical with";

$\doteq$, read "approaches."

Signs of Aggregation. — When several quantities are to be treated as a single one, they are enclosed by parentheses ( ), brackets [ ], or braces { }, or a line is drawn over them, called a vinculum, $\overline{\quad}$.

Signs of Quality. — These are

$+$, positive; $-$, negative; $|\ |$, absolute value.

The first two simply indicate opposite qualities; thus, if $+v$, or simply v, denote a velocity in one direction, then $-v$ denotes an equal velocity in the opposite direction; if $+t$ denote a temperature above zero, $-t$ denotes an equal temperature below zero. The third symbol is used to indicate that we are dealing simply with the numerical (absolute) magnitude of a quantity, without regard to its sign.

3. The Four Fundamental Operations. — These are, addition, subtraction, multiplication and division, indicated by the symbols $+$, $-$, $\times$, $\div$, respectively. It will suffice to recall the rules or laws in accordance with which these operations are to be performed. They are here given in the form of equations, and the student is asked to state each in words.

Laws of Addition.

1. If $a = b$ and $c = d$, then $a + c = b + d$.
2. If $a = b$ and $c \neq d$, then $a + c \neq b + d$.
3. $a + b = b + a$. (*Commutative law.*)
4. $(a + b) + c = a + (b + c)$. (*Associative law.*)

Laws of Subtraction. — (Subtraction defined by $(a-b)+b=a$.)

1. If $a = b$ and $c = d$, then $a - c = b - d$.
2. $(a - c) + b = (a + b) - c$.
3. $a + (b - c) = (a + b) - c$.
4. $(a + c) - b = (a - b) + c$.
5. $(a - c) - b = (a - b) - c$.
6. $a - (b + c) = (a - b) - c$.
7. $a - (b - c) = (a - b) + c$.

Laws of Multiplication.

1. If $a = b$ and $c = d$, then $ac = bd$.
2. If $a = b$ and $c \neq d$, then $ac \neq bd$.
3. $a \times b = b \times a$. (*Commutative law.*)
4. $a \times (b \times c) = (a \times b) \times c$. (*Associative law.*)
5. $(a + b - c) \times d = a \times d + b \times d - c \times d$. (*Distributive law.*)

Laws of Division. — (Division defined by $(a \div b) \times b = a$.)

1. If $a = b$ and $c = d$, then $a \div c = b \div d$, provided $c, d \neq 0$.
2. $(a \div b) \times c = (a \times c) \div b$, provided $b \neq 0$.
3. $a \times (b \div c) = (a \times b) \div c$, provided $c \neq 0$.
4. $(a \div b) \div c = (a \div c) \div b$, provided $b, c \neq 0$.
5. $a \div (b \div c) = (a \div b) \times c$, provided $b, c \neq 0$.

Some Working Rules. — The sign before a parenthesis may be changed if the sign of each of the terms enclosed is changed also.

When several quantities are to be subtracted, change their signs and add them.

Division may be expressed as a multiplication of dividend by reciprocal of divisor.

The sign of a product will be $+$ or $-$, according as there are an even or an odd number of negative factors.

4. Rational Numbers. — All positive integers can be formed by adding $+1$ to itself a sufficient number of times. Through the operation of subtraction, negative integers are introduced. By performing the operations of addition, subtraction and multiplication on the system of positive and negative integers, no new numbers are formed. Division, however, does introduce a new class of numbers, namely fractions, positive or negative, formed of the quotient of two integers.

All numbers, positive or negative, which are formed of the quotient of two integers, are called *rational numbers*. They can be obtained from $+1$ by means of the four fundamental operations.

Rational Expressions. — Let there be given certain quantities, $a, b, \ldots x, y, \ldots.$ Any expression which can be built up from these quantities by means of the four fundamental operations is called a *rational expression* (or function) in terms of the quantities involved.

5. Zero. — *Zero is defined as that number which may be added to any quantity without changing the value of the quantity.* As an equation, the definition is

$$a + 0 = a.$$

Since
$$(a - 0) + 0 = a,$$

it also follows that

$$a - 0 = a.$$

6. The operation of *division by zero is excluded,* because, whatever be the number a, there is no number which represents $a \div 0$. The reason for this we proceed to consider. In the first place, 0 must be less in absolute value than any assignable number, however small. For if this were not the case, we would have $a + 0 \neq a$. Now consider the quotient $\frac{a}{b}$, and suppose a to be fixed, and b to be taken smaller and smaller. As b tends toward zero, the quotient $\frac{a}{b}$ increases without limit and becomes larger than any assignable number. But as b approaches zero, $\frac{a}{b}$ takes the form $\frac{a}{0}$ and at the same time increases without limit so that no value can be assigned to this form.

Example. Let $x = 1.$

Then $x = x^2$

and $1 - x = 1 - x^2 = (1 + x)(1 - x).$

Dividing by $1 - x$, we have

$$1 = 1 + x.$$

Therefore $1 = 2$, since $x = 1$.

We are led to this fallacy by dividing by zero in the form of $1 - x$. Since we assumed $x = 1$, therefore $1 - x = 0$, and hence division by $1 - x$ must be excluded in this problem.

In any expression involving fractions, those cases in which the denominator of any fraction vanishes must be treated as exceptional and especially considered.

If, in a product, a factor approaches zero, while the other factors have any assigned values, then the product approaches zero. This is expressed by the equation

$$a \times 0 = 0.$$

7. Infinity. — A quantity which increases without limit is said to become infinite. When $b \doteq 0$ ("b approaches zero"), if a is any fixed number, $\frac{a}{b}$ increases without limit. Such quantities, which are larger than any assignable number, are all indicated by the same symbol, ∞ (read "infinity"). As an example, consider the law of gases, pressure times volume is constant, or

$$pv = c, \quad \text{or} \quad p = \frac{c}{v}.$$

When v is very small (relative to the constant c), p will be very large, and as v becomes still smaller, p must increase. We can choose v so small that p will exceed any assignable quantity, or p becomes ∞ when $v \doteq 0$. This is often indicated by $\lim_{v=0} p = \infty$ (read "the limit of p is infinity, when v approaches zero").

We are thus led to write the equation,

$$\frac{a}{0} = \infty, \quad \text{when} \quad a \neq 0.$$

This is not a proper equation, but simply an abbreviation for the statement, "A fraction whose numerator is not zero, and whose denominator approaches zero, becomes larger than any assignable quantity."

Since a quantity which increases without limit can be made as large as we please after being increased or diminished, multiplied or divided by any number, we have

$$\infty + a = \infty, \quad \infty - a = \infty; \quad \infty \times a = \infty, \quad \infty \div a = \infty.$$

8. Powers. — For brevity we put $a \times a = a^2$, $a \times a \times a = a^3$, and $a \times a \times a \ldots$ to n factors $= a^n$. The quantity a^n is called the *nth power* of a. The number n is called the **exponent** and a the **base** of the quantity a^n.

9. Some Important Relations. — The following equations and statements should be verified carefully and committed to memory:

1. $(a + b)^2 = a^2 + 2ab + b^2$.
2. $(a - b)^2 = a^2 - 2ab + b^2$.
3. $a^2 - b^2 = (a + b)(a - b)$.
4. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
5. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
6. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$.
7. The square of any polynomial equals the sum of the squares of the separate terms plus twice the product of each term by each following term.
8. $a^n - b^n$ is divisible by $(a + b)$ and $(a - b)$ when n is even.
9. $a^n - b^n$ is divisible by $(a - b)$, not by $(a + b)$, when n is odd.
10. $a^n + b^n$ is divisible by $(a + b)$, not by $(a - b)$, when n is odd.
11. $a^n + b^n$ is not divisible by $(a + b)$ or $(a - b)$, when n is even.

10. Exercises. — Simplify, by removal of parentheses and collection of like terms:

1. $(\frac{3}{4}a - \frac{1}{2}) + (\frac{1}{4} - \frac{1}{2}a)$.
2. $(\frac{1}{5}a^2b - \frac{3}{4}ab^2) + (\frac{1}{2}a^2b + \frac{7}{8}ab^2)$.
3. $(0.8a^2 - 3.47ab - 17.25ac) - (\frac{3}{4}a^2 - 0.47ab - 12\frac{5}{8}ac)$.
4. $(\frac{5}{2}x^2 + 3ax - \frac{7}{3}a^2) - (2x^2 - ax - \frac{1}{2}a^2)$.
5. $44x + \{[48y - (6z + 3y - 7x) + 4z] - [48y - 8x + 2z - (4x + y)]\}$.
6. $6a - \{4a - [8b - \overline{2a + b}] + (3b - 4a)\}$.

Perform the operations indicated in the following exercises and simplify the results when possible:

7. $\frac{2}{3}a^2c(\frac{3}{2}b^2 - 4c^3 + \frac{2}{5}ad^2 - 3)$.
8. $3xy^2(x^3 - 3x^2y + 3xy^2 - 2)$.
9. $0.6ac^2d^4(2a^2b - 3cd^3 + \frac{1}{2}ac^3 - 5)$.
10. $3\frac{1}{2}a^2bc(6a^4 - 4b^2 + 2ab^3 - 3c^2)$.
11. $(x^3 - 2x + 1)(x^3 - 3x + 2)$.
12. $(3a^3b - 2a^2b^2 + ab^3)(2a^2 - ab - 5b^2)$.
13. $(x^4 - 7x^2y^2 + 6xy^3 - y^4)(x^3 - 2xy^2 + y^3)$.
14. $(a + b)^3 + (a - b)^3$.
15. $(\frac{1}{2}a + \frac{1}{3})^3 - (\frac{1}{2}a - \frac{1}{3})^3$.
16. $(x^2 + 1 - y^2 + 2y)(x^2 + 1 + y^2 - 2y)$.
17. $[x^2 + (a + b)x + ab][x^2 - (a + b)x + ab]$.
18. $[(x + a)^2 - ax][(x - a)^2 + ax]$.
19. $a(a + 1)(a + 2) - (a - 1)(a - 2)(a - 3)$
20. $[x(y - 1) - y(x - 1)][(x + y)^2 - (x - y)^2]$.
21. $31\frac{1}{5}m^4np^5 \div -10\frac{2}{5}m^2np^4$.
22. $a^2bc^7 \div \frac{5}{11}a^4b^2c^8$.
23. $\frac{15}{16}x^4y^5 \div -\frac{31}{32}x^3y^3$.
24. $3a^2(b + x)^3 \div 6a^3(b + x)^2$.
25. $1.75x^3(x^2 - 1)^4 \div 25x^5(1 - x^2)^2$.
26. $(8a^5b - 24a^4b^3 + 16a^7b^8) \div -8a^4b$.
27. $(8x^3y - \frac{5}{6}xy^4 - \frac{1}{3}y^5 + 2y^2) \div -\frac{4}{5}x^2y^3$.
28. $x^5(a^2 + b^2) - 2x^4(a^2 + b^2)^3 \div x^3(a^2 + b^2)$.
29. $(6a^3x - 17a^2x^2 + 14ax^3 - 3x^4) \div (2a - 3x)$.
30. $(4y^4 - 18y^3 + 22y^2 - 7y + 5) \div (2y - 5)$.
31. $[2x^3 + 7x^2y - 9y^2(x + y)] \div (2x - 3y)$.
32. $(-\frac{1}{8}d^5 + \frac{5}{6}d^4 - \frac{41}{24}d^3 + d^2) \div (-\frac{3}{4}d^2 + 2d)$.
33. $(\frac{9}{16}a^4 - \frac{7}{8}a^3b + \frac{19}{36}a^2b^2 + \frac{1}{6}ab^3) \div (\frac{3}{2}a + \frac{1}{3}b)$.
34. $(\frac{1}{14}m^3 + \frac{1}{24}m^2n - \frac{25}{12}mn^2 + \frac{14}{9}n^3) \div (\frac{1}{2}m - \frac{7}{3}n)$.
35. $(x^5 - \frac{23}{30}x^4 + \frac{31}{10}x^5 - \frac{7}{3}x^2 - \frac{181}{18}x + \frac{5}{3}) \div (x^2 - \frac{1}{6}x + 5)$.
36. $(2a^3 - 16a + 6) \div (a + 3)$.
37. $(4x^4 - x^2y^2 + 6xy^3 - 9y^4) \div (2x^2 - xy + 3y^2)$.
38. $(x^4 + 4x^2y^2 - 32y^4) \div (x - 2y)$.
39. $(a^5 - 5a^3b^2 - 5a^2b^3 + b^5) \div (a^2 - 3ab + b^2)$.
40. $(x^3 - 8y^3) \div (x - 2y)$.
41. $(\frac{1}{81}x^4 - 9y^2) \div (\frac{1}{9}x^2 + 3y)$.

42. $(27\,a^3b^3 + 64\,x^3y^3) \div (3\,ab + 4\,xy)$.
43. $(a^3b^3 + c^3) \div (a^2b^2 - abc + c^2)$.
44. $(u^5 - 32\,v^5) \div (u - 2\,v)$.
45. $(a - b + c - d)^2$.
46. $(x - \frac{1}{2}\,y - 2\,u + w)^2$.

11. Factoring. — To *factor an expression* is to find two or more quantities whose product equals the given expression. When two or more expressions contain the same factor, it is called their *common factor*.

We shall illustrate the methods commonly used in factoring given expressions by means of some typical examples.

(*a*) *Expressions, each of whose terms contains a common factor.*

Example. $\frac{1}{6}\,x^3y^2z^4 + \frac{1}{9}\,x^2y^2z - \frac{1}{21}\,x^4y^3z^2 = \frac{1}{3}\,x^2y^2z\,(\frac{1}{2}\,xz^2 + \frac{1}{3} - \frac{1}{7}\,x^2yz)$.

(*b*) *Expressions whose terms can be grouped, so that each group contains the same factor.*

Example.
$$\begin{aligned} x^3 - 7\,x^2y + 14\,xy^2 - 8\,y^3 &= (x^3 - 8\,y^3) - (7\,x^2y - 14\,xy^2) \\ &= (x - 2\,y)\,(x^2 + 2\,xy + 4\,y^2) - 7\,xy\,(x - 2\,y) \\ &= (x - 2\,y)\,(x^2 - 5\,xy + 4\,y^2) \\ &= (x - 2\,y)\,(x - y)\,(x - 4\,y). \end{aligned}$$

(*c*) *Trinomials of the form* $ax^2 + bx + c$.

Let h, k be a pair of factors whose product is a, and m, n a pair whose product is c. Arrange these four factors as in the adjacent schemes $\begin{smallmatrix} h & & n \\ & \times & \\ k & & m \end{smallmatrix}$ $\begin{smallmatrix} h & & m \\ & \times & \\ k & & n \end{smallmatrix}$ and form the cross-products as indicated. The sum of the cross-products must equal b. If this is true in the first scheme, the factors are $(hx + n)\,(kx + m)$; in the second, the factors are $(hx + m)\,(kx + n)$.

Example. $12\,x^2 - 7\,x - 10$.

Here h, k may be one of the pairs of numbers 1, 12, or 2, 6, or 3, 4, both numbers to be taken with the same sign. The numbers m, n may be -1, 10, or $+1$, -10, or -2, 5, or $+2$, -5. By trial we find that h, k must be 3, 4, and m, n must be 2, -5. The factors are therefore $(3\,x + 2)\,(4\,x - 5)$.

To find the factors of $12\,x^2 - 7\,xy - 10\,y^2$, we would proceed as above and obtain $(3\,x + 2\,y)\,(4\,x - 5\,y)$.

(*d*) *Expressions which can be written as the difference of the squares of two quantities.*

The factors are the sum and the difference of the two quantities respectively.

Example.
$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2\,a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + ab + b^2)\,(a^2 - ab + b^2). \end{aligned}$$

(*e*) *Expressions of the form* $P^2 + 2PQ + Q^2$, *where* P *and* Q *are monomials or polynomials.*

The expression is then the product of two factors each equal to $(P + Q)$, and is therefore $(P + Q)^2$.

Example.
$$\begin{aligned} x^2 + y^2 - 2xy - 4ax + 4ay + 4a^2 &= (x - y)^2 - 4a(x - y) + 4a^2 \\ &= (x - y - 2a)^2. \end{aligned}$$

(*f*) **Factor Theorem.** — If a polynomial in x reduces to zero when x is replaced by h, the polynomial contains the factor $(x - h)$.

Proof: Let the polynomial be

$$P \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n.$$

Putting h for x, we have by hypothesis

$$a_0h^n + a_1h^{n-1} + a_2h^{n-2} + \cdots + a_{n-1}h + a_n = 0.$$

Therefore by subtraction,

$$P = a_0(x^n - h^n) + a_1(x^{n-1} - h^{n-1}) + a_2(x^{n-2} - h^{n-2}) + \cdots + a_{n-1}(x - h).$$

But each term of the right member of the last equation contains the factor $x - h$. (See 8 and 9 of **9**.) Hence P is divisible by $(x - h)$.

Example. Factor $x^3 + 3x^2 - 4x - 12$.

If this is the product of three factors $(x - h)(x - k)(x - l)$, then evidently $hkl = 12$. Hence we substitute in the given polynomial the factors of 12, and find that it vanishes when $x = 2$, $x = -2$, and $x = -3$. Hence the factors are $(x - 2)(x + 2)(x + 3)$.

12. Exercises. — Factor:

1. $\dfrac{x^2}{2a} - \dfrac{xy}{8a^3} - \dfrac{x}{4a^2}$.

2. $x^2 - 2ax + a^2 - y^2$.

3. $3x^2 - 4x + 1$.

4. $15x^2 - 7x - 2$.

5. $6x^2 + 19xy - 7y^2$.

6. $x^2 - 2x - 24$.

7. $8x^4 - 27xy^3$

8. $27x^4 + 8xy^3$.

9. $x^4 - 13x^2 + 36$.

10. $4a^4 - 5a^2 + 1$.

11. $x^2 - x - 110$.

12. $7 + 10x + 3x^2$.

13. $x^2 - 10a^2x + 9a^4$.

14. $(a + b)^4 - 1$.

15. $x^2y^2 - 3xyz - 10z^2$.

16. $2 + 7x - 15x^2$.

17. $x^3 - 64x - x^2 + 64$.

18. $a^8 + 1$.

19. $x^6 - 1$.

20. $(a + b)^3 + 1$.

21. $(x^3 + y^3) - (x + y)^3$.
22. $a^4 + b^4 - c^4 - d^4 + 2a^2b^2 - 2c^2d^2$.
23. $a^2c^2 + acd + abc + bd$.
24. $1 - a^2x^2 - b^2y^2 + 2abxy$.
25. $x^4y - x^2y^3 - x^3y^2 + xy^4$.
26. $a^8 - 82a^4 + 81$.
27. $x^4y^2 - 17x^2y - 110$.
28. $(a^2 + 3)^2 - 36a^2$.
29. $x^3 + 9x^2 + 16x +$
30. $4x^4 - 8x^3 - x^2 +$.

13. Highest Common Factor. — The *highest common factor* (*H. C. F.*) of two or more polynomials is the polynomial of highest degree that will divide them all without a remainder.

When each of the given polynomials can be factored by inspection, the *H. C. F.* is easily determined from their common factors.

Example. The *H. C. F.* of $32(x-1)^2(x+1)^3(x^2+1)$ and $24(x-1)^3(x+1)^2(x^2+1)^2$ is $8(x-1)^2(x+1)^2(x^2+1)$.

When the given polynomials cannot be readily factored, we use a method like that of arithmetic.

Let the given polynomials be P_1 and P_2 and let Q be the quotient and R the remainder when P_1 is divided by P_2. Then

$$P_1 = P_2Q + R.$$

Hence any factor common to P_1 and P_2 is also a factor of R. Hence it is a common factor of P_2 and R. Divide P_2 by R, obtaining

$$P_2 = RQ_1 + R_1.$$

Hence a common factor of P_2 and R is also a factor of R_1. Dividing R by R_1, we obtain

$$R = R_1Q_2 + R_2,$$

and the common factor must be present in R_2, and so on.

Rule. — If at any step there is no remainder, the last divisor is the required *H. C. F.*

14. Least Common Multiple. — The *least common multiple* (*L. C. M.*) of two or more polynomials is the polynomial of lowest degree that is exactly divisible by each of them.

When the given polynomials can be easily factored by inspection, form the product of all the types of factors present in any of them, taking each factor the greatest number of times that it occurs in any of the given expressions; this product is their *L. C. M.*

When the given polynomials cannot readily be factored, their *L. C. M.* is obtained by use of the following theorem:

The product of the H. C. F. and L. C. M. of two polynomials equals the product of the polynomials.

Proof: Let F be the *H. C. F.*, and M the *L. C. M.* of the two polynomials P_1 and P_2. Also let

$$\frac{P_1}{F} = Q_1 \text{ and } \frac{P_2}{F} = Q_2;$$

then $\quad P_1 = FQ_1$ and $P_2 = FQ_2$.

Hence $\quad P_1P_2 = F \times FQ_1Q_2$.

Since F contains all factors common to P_1 and P_2, Q_1 and Q_2 have no common factor, and the product FQ_1Q_2 contains all the factors of the types present in both P_1 and P_2.

$$\therefore \quad M = FQ_1Q_2 = \frac{P_1P_2}{F}; \quad \text{or,} \quad MF = P_1P_2.$$

Rule. — To find the *L. C. M.* of two polynomials, divide their product by their *H. C. F.*

To find the *L. C. M.* of more than two polynomials, find the *L. C. M.* of two of them, then the *L. C. M.* of this and a third one of the polynomials, and so on.

15. Exercises. — Find the *H. C. F.* of

1. $6(x+1)^3$ and $9(x^2-1)$.
2. $a^6 - b^6$ and $a^4 - b^4$.
3. $12(x^2+y^2)^2$ and $8(x^4-y^4)$.
4. $u^5 - v^5$ and $u^2 - v^2$.
5. $(a^2x - ax^2)^2$ and $ax(a^2-x^2)^2$.
6. $27(a^4 - b^4)$ and $18(a+b)^2$.
7. $(24a^2 + 36ab - 48ac)$ and $(30a^3 + 45a^2b - 60a^2c)$.
8. $125x^3 - 1$ and $35x^2 - 7x + 5ax - a$.
9. $4x^2 - 12xy + 9y^2$ and $4x^2 - 9y^2$.
10. $x^2 + 2x - 120$ and $x^2 - 2x - 80$.
11. $12x^2 - 17ax + 6a^2$ and $9x^2 + 6ax - 8a^2$.
12. $x^3 + 4x^2 - 5x$ and $x^3 - 6x + 5$.
13. $x^3 + 3x^2 + 7x + 21$ and $2x^4 + 19x^2 + 35$.
14. $a^4 + 7a^3 + 7a^2 - 15a$ and $a^3 - 2a^2 - 13a + 110$.
15. $20x^4 + x^2 - 1$ and $75x^4 + 15x^3 - 3x - 3$.
16. $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4$ and $3x^3 - 7ax^2 + 3a^2x - 7a^3$.
17. $x^4 - y^4$, $x^3 + y^3$, and $x^5 + y^5$.
18. $x^2 - 2a^2 - ax$, $x^2 - 6a^2 + ax$, and $x^2 - 8a^2 + 2ax$.
19. $a^4 + a^2b^2 + b^4$, $a^4 + ab^3$, and $a^3b + b^4$.
20. $3x^3 - 7x^2y + 5xy^2 - y^3$, $x^2y + 3xy^2 - 3x^3 - y^3$, and $3x^3 + 5x^2y + xy^2 - y^3$.

Find the *L. C. M.* of:

21. $8a^2x^2y^3$ and $12abx^2y^2$.
22. $4ax^3y^2$, $6a^2xy^3$, and $18a^3x^2y$.
23. $a^2 - b^2$ and $(a - b)^2$.
24. $a^2bx - ab^2y$ and $abx + b^2y$.
25. $x^2 - 3x - 4$ and $x^2 - x - 12$.
26. $x^2 - 1$ and $x^2 + 4x + 3$.
27. $6x^2 + 5x - 6$ and $6x^2 - 13x + 6$.
28. $12x^2 + 5x - 3$ and $6x^3 + x^2 - x$.
29. $12x^2 - 17ax + 6a^2$ and $9x^2 + 6ax - 8a^2$.
30. $a^3 - 9a^2 + 23a - 15$ and $a^2 - 8a + 7$.
31. $m^3 + 2m^2n - mn^2 - 2n^3$ and $m^3 - 2m^2n - mn^2 + 2n^3$.
32. $x^2 - y^2$, $(x - y)^2$, and $x + y$.
33. $x^2 + 3x + 2$, $x^2 + 4x + 3$, and $x^2 + 5x + 6$.
34. $x^2 + 5x + 10$, $x^3 - 19x - 30$, and $x^3 - 15x - 50$.
35. $x^2 + 2x - 3$, $x^3 + 3x^2 - x - 3$, and $x^3 + 4x^2 + x - 6$.
36. $6x^2 - 13x + 6$, $6x^2 + 5x - 6$, and $9x^2 - 4$.
37. $x^2 - 1$, $x^2 + 1$, and $x^3 + 1$.
38. $x^2 + 1$, $x^4 - 1$, and $x^6 - 1$.
39. $a^3 - b^3$, $a^9 - b^9$, and $a^6 - b^6$.
40. $x^2 - y^2$, $x^3 + y^3$, $x^3 - y^3$, and $x^6 + y^6$.

16. Fractions. — An algebraic fraction is the indicated quotient of two algebraic expressions. It is written in the form $\frac{N}{D}$, N being called the numerator and D the denominator.

When N and D have a common factor F, so that we may put

$$N = N_1F \text{ and } D = D_1F,$$

then the fraction may be simplified as follows:

$$\frac{N}{D} = \frac{N_1F}{D_1F} = \frac{N_1}{D_1}.$$

When all factors common to N and D have been removed in this way, the fraction is said to be *reduced to its lowest terms.*

When the common factors of N and D are not obvious on inspection, find the *H. C. F.* of N and D, and remove it as above.

17. Sign of a Fraction. — By the rules for division we have,

$$\frac{N}{D} = -\frac{-N}{D} = -\frac{N}{-D} = \frac{-N}{-D}.$$

Hence the rules: *Changing the sign of either numerator or denominator changes the sign of the fraction.*

Changing the signs of both numerator and denominator does not affect the sign of the fraction.

The sign of a fraction may be changed either by changing the sign standing before the fraction, or by changing the sign of the numerator or of the denominator.

18. An **integral expression** is one whose literal parts are free from fractions.

A **mixed expression** is one formed from the sum of an integral part and one or more fractions.

A **complex fraction** is one whose numerator, or denominator, or both are fractions or mixed expressions.

Every mixed expression and every complex fraction can be reduced to a simple fraction (or to an integral expression).

For, two or more simple fractions can be reduced to a *common denominator* and then combined into a single fraction by writing the sum of the numerators over the common denominator. For this purpose the simplest common denominator is the *L. C. M.* of the separate denominators. This is called the **least common denominator** of the fractions considered. In this manner we reduce

$$\frac{N_1}{D_1} + \frac{N_2}{D_2} + \cdots \text{ to } \frac{N}{D}.$$

A mixed expression is reduced by the formula

$$P + \frac{N}{D} = \frac{PD + N}{D}.$$

Finally, a complex fraction is reduced by first reducing its numerator and denominator separately to simple fractions. The reduction is then completed by the formula,

$$\frac{\frac{N}{D}}{\frac{N'}{D'}} = \frac{N}{D} \times \frac{D'}{N'} = \frac{ND'}{N'D}.$$

Examples.

1. Simplify $$\frac{2}{\frac{1}{x} - \frac{1}{y}} + \frac{y}{1 - \frac{y}{x}} - \frac{x}{1 - \frac{x}{y}}.$$

First reduce each fraction to a simple fraction, thus:

$$\frac{2}{\frac{1}{x} - \frac{1}{y}} = \frac{2}{\frac{y - x}{xy}} = \frac{2xy}{y - x},$$

$$\frac{y}{1 - \frac{y}{x}} = \frac{y}{\frac{x - y}{x}} = \frac{xy}{x - y},$$

$$-\frac{x}{1-\frac{x}{y}} = -\frac{x}{\frac{y-x}{y}} = -\frac{xy}{y-x} = \frac{xy}{x-y}.$$

Reducing to the common denominator $x - y$, we have

$$\frac{2\,xy}{y-x} + \frac{xy}{x-y} + \frac{xy}{x-y} = \frac{-2\,xy + xy + xy}{x-y} = 0 \text{ (provided } x \neq y).$$

2. $$x^3 - \frac{x^2}{x + \frac{1-x^4}{x-\frac{1}{x}}} = x^3 - \frac{x^2}{x + \frac{1-x^4}{\frac{x^2-1}{x}}} = x^3 - \frac{x^2}{x + \frac{x\,(1-x^4)}{x^2-1}}$$

$$= x^3 - \frac{x^2}{x - x\,(1+x^2)} = x^3 - \frac{x^2}{-x^3} = x^3 + \frac{1}{x} = \frac{x^4+1}{x}.$$

19. Exercises. — Reduce to simple fractions or to integral expressions:

1. $\left(\frac{a+x}{a-x} - \frac{a-x}{a+x}\right)(a^2 - x^2).$

2. $\frac{a^3-b^3}{a^3+b^3} \div \frac{a^2-b^2}{2\,ab}.$

3. $\frac{a^4-b^4}{x^3-y^3} \times \frac{x^2+xy+y^2}{a^2+b^2}.$

4. $\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{a}{x}+\frac{b}{y}\right).$

5. $\left(\frac{x^5}{y^5}-\frac{y^5}{x^5}\right) \div \left(\frac{x}{y}-\frac{y}{x}\right).$

6. $\frac{x^{12}+y^{12}}{x^{12}-y^{12}} \div \frac{x^4+y^4}{x^8-y^8}.$

7. $\frac{x^2+7\,x+12}{x^2-x-12} \div \frac{x^2+6x+8}{x^2-2x-8}.$

8. $\frac{\frac{x+1}{x}+\frac{y+1}{y}}{\frac{1}{x}-\frac{1}{y}}.$

9. $1 + \frac{1}{x+\frac{1}{x}}.$

10. $\frac{\frac{x}{x+y}+\frac{y}{x-y}}{\frac{x}{x-y}-\frac{y}{x+y}}.$

11. $\frac{x^2+y^2}{x^2-y^2} \div \frac{3\,x^2+3\,y^2}{x+y}.$

12. $\frac{45\,(x-y)}{32\,(z+y)} \div \frac{27\,(x-y)^2}{128\,b\,(z+y)^2}.$

13. $\frac{a^2-4\,x^2}{a^2+4\,ax} \div \frac{a^2-2\,ax}{ax+4\,x^2}.$

14. $\frac{a^2+ab}{a^2+b^2} \div \frac{ab\,(a+b)^2}{a^4-b^4}.$

15. $\frac{u^3-v^3}{u^2v^2-u^4} \div \frac{u^2+uv+v^2}{uv^2+v^3}.$

16. $\frac{p^2+3\,p+9}{p^4-3\,p^2+9} \div \frac{p^3-27}{p^6+27}.$

17. $\frac{x^6-y^6}{(x-y)^2} \div \frac{x^2+xy+y^2}{x-y}.$

18. $\frac{\frac{x-y}{x+y}-\frac{x^3-y^3}{x^3+y^3}}{\frac{x+y}{x-y}+\frac{x^2+y^2}{x^2-y^2}}.$

19. $\frac{1}{x+\frac{1}{1+\frac{1+y}{3-y}}}.$

20. $\frac{1}{x\,(x-1)} + \frac{2}{1-x^2} + \frac{1}{x\,(x+1)}.$

21. $\frac{1}{x^2-5\,x+6} - \frac{2}{x^2-4\,x+3} + \frac{1}{x^2-3\,x+2}.$

22. $\frac{u}{u+v}+\frac{u}{u-v}+\frac{2u^2}{u^2+v^2}$.

23. $\frac{x-1}{x^2-5x+6}-\frac{2(x-2)}{x^2-4x+3}+\frac{x-3}{x^2-3x+2}$.

24. $\frac{7+3x^2}{4-x^2}-\frac{5-2x^2}{4+4x+x^2}-\frac{3-2x+x^2}{4-4x+x^2}$.

25. $\frac{1-2x}{3(x^2-x+1)}-\frac{2x-3}{2(x^2+1)}+\frac{1}{6(x+1)}$.

26. $\left(\frac{a}{bc}-\frac{b}{ac}-\frac{c}{ab}-\frac{2}{a}\right)\left(1-\frac{2c}{a+b+c}\right)$.

27. $\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}-\frac{x}{a}-\frac{a}{x}+1\right)\left(\frac{x}{a}-\frac{a}{x}\right)$.

28. $\left(\frac{x}{a}-\frac{a}{x}+\frac{y}{b}-\frac{b}{y}\right)\left(\frac{x}{a}-\frac{a}{x}-\frac{y}{b}+\frac{b}{y}\right)$.

29. $\left(1-\frac{7x}{11y}\right)\left(\frac{7x}{11y}+\frac{49x^2}{121y^2}+\frac{343x^3}{1331y^3}\right)$.

30. $\left(1\frac{2}{3}\frac{a}{b}-4\frac{5}{6}\frac{b}{a}\right)\left(7\frac{8}{9}\frac{b}{a}-10\frac{1}{12}\frac{a}{b}\right)-\left(\frac{7a}{9b}+\frac{5b}{12a}\right)\left(\frac{7a}{9b}-\frac{5b}{12a}\right)$.

31. $\left(\frac{ab}{3c^2}-\frac{3bc}{5a^2}\right)\left(\frac{5ac}{7b^2}-\frac{7ab}{9c^2}\right)\left(\frac{3b}{5a^2}-\frac{ab}{3c^2}\right)$.

32. $\left(\frac{4x^3y^2}{5a^3}-\frac{3x^2y^3}{2a^2b}+\frac{2xy^4}{3ab^2}-\frac{y^5}{b^3}\right)\left(\frac{2x^2y}{3a^2}-\frac{3xy^2}{5ab}-\frac{3y^3}{2b^2}\right)$.

CHAPTER II

Involution. Evolution. Theory of Exponents. Surds and Imaginaries

20. Involution is the operation of raising a quantity to an indicated power.

The symbol a^n represents $a \times a \times a \ldots$ to n factors (**8**), n being a positive integer. Hence, if m be a second positive integer, we have by cancellation,

(1) $$\frac{a^n}{a^m} = a^{n-m} \text{ when } n > m;$$

(2) $$\frac{a^n}{a^m} = \frac{1}{a^{m-n}} \text{ when } n < m.$$

Negative Exponent. — We now *define* the symbol a^{-n} to be

$$a^{-n} \equiv \frac{1}{a^n} = \frac{1}{a \times a \times a \ldots \text{ to } n \text{ factors}}.$$

Then $$\frac{1}{a^{m-n}} = a^{-(m-n)} = a^{n-m}.$$

We may now write,

(3) $$\frac{a^n}{a^m} = a^{n-m},$$

whether n is greater or less than m. Hence by the introduction of the *negative exponent*, the two equations (1), (2), may be written as a single equation, (3).

We now easily verify the following *rules for operating with integral exponents, positive or negative.*

I. $a^{-n} = \dfrac{1}{a^n}.$ IV. $(a^m)^n = a^{mn}.$

II. $a^n \times a^m = a^{n+m}.$ V. $(ab)^n = a^n b^n.$

III. $a^n \div a^m = a^{n-m}.$ VI. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}.$

21. Exercises.

1. State the above rules in words.
2. Verify the above rules by means of the definitions for a^n and a^{-n}.
3. Show that rule II contains rule III.
4. Show that rule V contains rule VI.

Perform the operations indicated in the following exercises, and express the results in forms free from fractions:

5. $\left(\frac{a^2b^3}{c^4d^5}\right)^2$.
6. $\left(\frac{3\,xy^3}{4\,m^2n^5}\right)^3$.
7. $\frac{(ab)^6}{(a^2b^3)^3}$.
8. $[(ax)^{3m+4n}]^{5m-6n}$.
9. $\left(\frac{m^5n^3}{p^2q^2}\right)^3\left(\frac{mq^3}{n}\right)^4$.
10. $\left(\frac{a^4b^3}{c^2x^3}\right)^5 \div \left(\frac{ax^3}{bc^4}\right)^4$.

Zero Exponent. — If in rule III we put $n = m$, we get

$$a^n \div a^n = a^0.$$

But $a^n \div a^n = 1$. Therefore we *define* the symbol a^0 by the equation $a^0 = 1$. Then III is true for all integral values of n and m, equal or unequal. Hence we add to the above rules:

VII. $a^0 = 1$.

22. The *nth root of a quantity a* (symbol $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$) is a quantity whose nth power is equal to a.

Evolution is the operation of finding the indicated root of a quantity.

By definition, we have

$$\sqrt[n]{a} \times \sqrt[n]{a} \times \sqrt[n]{a} \;\ldots\; \text{to } n \text{ factors} = (\sqrt[n]{a})^n = a,$$

or

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \;\ldots\; \text{to } n \text{ factors} = \left(a^{\frac{1}{n}}\right)^n = a.$$

The last equation will be covered by rule IV (**20**) if we extend that rule to the case where m is the reciprocal of a positive integer. We now extend rules I–VI and *assume* that m and n may be not only positive or negative integers or zero, but also the reciprocals of positive or negative integers.

If we let $n = \frac{1}{r}$ and $m = \frac{1}{s}$, r and s being integers, we have

I′. $a^{-\frac{1}{r}} = \dfrac{1}{a^{\frac{1}{r}}}$.

II′. $a^{\frac{1}{r}} \times a^{\frac{1}{s}} = a^{\frac{1}{r}+\frac{1}{s}}$.

III′. $a^{\frac{1}{r}} \div a^{\frac{1}{s}} = a^{\frac{1}{r}-\frac{1}{s}}$.

IV′. $\left(a^{\frac{1}{s}}\right)^{\frac{1}{r}} = a^{\frac{1}{rs}}$.

V′. $(ab)^{\frac{1}{r}} = a^{\frac{1}{r}} b^{\frac{1}{r}}$.

VI′. $\left(\dfrac{a}{b}\right)^{\frac{1}{r}} = \dfrac{a^{\frac{1}{r}}}{b^{\frac{1}{r}}}$.

These equations **define** the rules governing operations involving roots.

Exercise. State the above rules in words. What is the meaning of a negative root?

23. Rational Exponent. — By the preceding laws we now have a meaning assigned to the symbol a^n when n is any rational number (**4**). For, if $n = p \div q$, p and q being integers, we have

$$a^n = a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = (a^p)^{\frac{1}{q}};$$

that is, $a^{\frac{p}{q}}$ means the pth power of the qth root of a, or the qth root of the pth power. In a fractional exponent, the numerator is the index of the power, the denominator the index of the root.

By combining rules I–VI and I′–VI′, we see that the former set of rules holds when m and n are any rational numbers. Hence *we adopt the rules of* (**20**) *as the rules governing quantities affected with* **rational exponents.**

24. Irrational Numbers. — By the operation of evolution we are led to numbers which cannot be produced from integers by means of the four fundamental operations. Thus if we attempt to calculate $\sqrt{2}$ we are led to a non-terminating decimal. To four decimals we have

$$1.4142 < \sqrt{2} < 1.4143,$$

or

$$\frac{14142}{10000} < \sqrt{2} < \frac{14143}{10000}.$$

We have here two rational numbers between which $\sqrt{2}$ lies. By going out to a sufficient number of decimals, we can obviously obtain two rational numbers containing $\sqrt{2}$ between them and differing from it by as little as we please. By taking successively 4, 5, 6, . . . decimals, proceeding as above and noting each time the smaller of the two rational numbers, we obtain a series or sequence of rational numbers which increase and approach $\sqrt{2}$; by noting each time the larger of the two numbers, we obtain a second sequence of rational numbers which decrease and also approach $\sqrt{2}$.

If on the other hand we consider the sequence of numbers

$$1, \frac{13}{10}, \frac{133}{100}, \frac{1333}{1000}, \ldots$$

these evidently approach the value $\frac{4}{3}$, which is a rational number.

The idea here indicated is used to define irrational numbers. Without going further into the subject here, we shall say that *an irrational number is one which can be represented to any degree of approximation, but not exactly as the quotient of two integers.* Such numbers may be produced in performing the operation of evolution on rational numbers.

Real Numbers. — The rational numbers, including all integers and quotients of integers, and the irrational numbers together constitute the class of *real numbers.*

Irrational Expressions. — We now extend the idea of irrationality to algebraic quantities in general by the following definition:

An algebraic expression is said to be ***irrational*** *when its parts are affected by other than the four fundamental operations.*

Hence any expression involving indicated roots is irrational. As examples, we have

$$\sqrt[5]{1+x^2};\ (x^2 - xy)^{-\frac{1}{3}} + (xy - y^2);\ \sqrt{\frac{1+2a+a^2}{1-a}}.$$

The last expression may be simplified. Thus,

$$\sqrt{\frac{1+2a+a^2}{1-a}} = \frac{\sqrt{1+2a+a^2}}{\sqrt{1-a}} = \frac{1+a}{\sqrt{1-a}}.$$

A **surd expression** is one involving an indicated root which cannot be exactly found.

A **surd number** is an indicated root of a number which cannot be exactly found.

25. Irrational Exponents. — What meaning shall we attach to the expression $2^{\sqrt{2}}$? Let $a_1, a_2, a_3, \ldots$ be a series of rational numbers approaching $\sqrt{2}$ in value. Then the quantity toward which the series of numbers $2^{a_1}, 2^{a_2}, 2^{a_3}, \ldots$ approaches is $2^{\sqrt{2}}$. Similarly we obtain a meaning for a^x, when x is irrational.

We now *define* a^x as a symbol subject to the following laws:

I. $a^{-x} = \frac{1}{a^x}$; IV. $(a^x)^y = a^{xy}$ (*not* a^{x^y});

II. $a^x a^y = a^{x+y}$; V. $(ab)^x = a^x b^x$;

III. $a^x \div a^y = a^{x-y}$; VI. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$;

provided that the symbols $a, b, x, y, a^x, b^x, a^y, b^y$ stand for real numbers.

26. Imaginary Numbers. — When $x^2 = 1$, we have obviously $x = \pm 1$. What is x when $x^2 = -1$? The answer cannot be a real number, since the square of every such number is positive. To obtain an answer to the question, we introduce a new number whose symbol is $\sqrt{-1}$, and which is defined as the quantity whose square is -1.

Since $\sqrt{-1}$ is not a real number, it is often called *imaginary* and denoted by i. Hence *the quantity* $i \equiv \sqrt{-1}$ *is defined by the equation* $i^2 = -1$.

We now *define* $\sqrt{-a}$ by the equation

I. $$\sqrt{-a} = i\sqrt{a}.$$

(This is in accordance with our rules for exponents, since

$$\sqrt{-a} = \sqrt{a \times -1} = \sqrt{a}\sqrt{-1} = i\sqrt{a}.)$$

Then the product $\sqrt{-a} \times \sqrt{-b}$ is determined by the equation,

II. $$\sqrt{-a} \times \sqrt{-b} = i\sqrt{a} \times i\sqrt{b} = i^2\sqrt{ab} = -\sqrt{ab}.$$

The results of the operations of algebra, applied to any number, are always expressible in the form $a + bi$, where a and b are real. Such a result may be considered as consisting of a real units and b imaginary units, $a \times 1 + b \times i$; it is called a *complex number*.

Two numbers of the forms $a + bi$ and $a - bi$ are called *conjugate complex numbers.*

When $a = 0$, the complex number $a + bi$ becomes bi called a *pure imaginary.*

The rules for operating with complex numbers, aside from II above, are considered in chapter 17.

Principal Root. — There are in general n distinct quantities, the nth power of each of which equals a given number a (see **259**). That is, a given number has in general n distinct nth roots. Thus,

the square of $+2$ or -2 is 4;
the cube of -2, $(1 + i\sqrt{3})$, or $(1 - i\sqrt{3})$ is -8;
the fourth power of $+2$, -2, $+2i$ or $-2i$ is 16.

The *principal root* of a number is its real positive root when one exists; if not, its real negative root; when all roots are imaginary, any one of them may be chosen as the principal root.

In this text the symbol for a root, $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$, will mean the principal root only.

Thus: $\sqrt{4} = 2$, not ± 2; if we wish to indicate both square roots, we always write $\pm\sqrt{a}$.

27. Reduction of Surds. — The expression $\sqrt[n]{a}$ is usually called a *radical*, $\sqrt{\ }$ being the *radical sign*, n the *index* of the radical and a the *radicand*. When the radicand is not a perfect nth power, the expression is a **surd**.

A surd is said to be in its *simplest form* when all factors of the radicand which are perfect powers of the same index as that of the radical have been taken out from the radical sign. Thus:

$$\sqrt[3]{\frac{8\,a^4b^5}{27\,c^6}} = \frac{2\,ab}{3\,c^2}\sqrt[3]{ab^2}.$$

Two surds are *similar* when they can be expressed with the same index and radicand. Otherwise they are *dissimilar*.

A *quadratic surd* is one whose index is 2.

28. *The sum, difference, product and quotient of two dissimilar quadratic surds are always surds.*

Proof: Let the surds be $\sqrt{a}$ and $\sqrt{b}$. Since they are dissimilar, neither ab nor $a \div b$ can be a perfect square. Hence the product or quotient of the two surds is a surd.

Further, let c be a rational number, and assume that

$$\sqrt{a} \pm \sqrt{b} = c.$$

Squaring, $$a \pm 2\sqrt{ab} + b = c,$$

or $$\pm 2\sqrt{ab} = c - a - b.$$

But a surd cannot equal a rational expression by definition. Hence the assumption is false, and the sum or difference of two surds is also a surd.

29. *Given a relation of the form $a + \sqrt{b} = c + \sqrt{d}$; then $a = c$ and $b = d$.*

For, on transposing, we have $\sqrt{b} - \sqrt{d} = c - a$; hence if $b \neq d$, we have a surd equal to a rational number, which is impossible. Therefore $b = d$. Hence also $a = c$.

30. *To rationalize the denominator of* $\dfrac{A}{\sqrt{a} + \sqrt{b}}$.

Rule.—Multiply both sides of the fraction by $\sqrt{a} - \sqrt{b}$.

For, $$\frac{A(\sqrt{a} + \sqrt{b})}{(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})} = \frac{A(\sqrt{a} + \sqrt{b})}{a - b}.$$

31. *To obtain the square root of $a + \sqrt{b}$.*

Assume that $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$. To find x and y.

Squaring, $$a + \sqrt{b} = x + y + 2\sqrt{xy} = x + y + \sqrt{4xy}.$$

Hence $$a = x + y \text{ and } b = 4xy \quad (\mathbf{29}).$$

Then $$a^2 - b = x^2 - 2xy + y^2 = (x - y)^2,$$

or $$\pm\sqrt{a^2 - b} = x - y.$$

But $$a = x + y.$$

Therefore $x = \frac{1}{2}(a \pm \sqrt{a^2 - b})$ and $y = \frac{1}{2}(a \mp \sqrt{a^2 - b})$.

32. *The index of a surd may be multiplied by any number if at the same time the radicand be raised to the power indicated by this number.*

For, $$\sqrt[n]{a} = a^{\frac{1}{n}} = a^{\frac{m}{mn}} = \sqrt[mn]{a^m}.$$

In combining surds by multiplication or division this rule is used to reduce them to surds with a common index. This is accomplished by writing all the surds as fractional exponents and then reducing the exponents to a common denominator.

33. Exercises.

Write the following with positive exponents and in simplest form:

1. $\dfrac{3a^0b^{-2}c^{-4}}{5a^{-1}b^{-3}c^{-5}}$.

2. $\dfrac{m^{2x+4}m^{-x-5}}{n^{x-3}}$.

3. $\left(\dfrac{a^{-3x}+b^{-2y}}{a^{-6x}-b^{-4y}}\right)^{-3}$.

4. $\left(\dfrac{a^{-2}+b^{-2}}{a^{-4}-b^{-4}}\right)^{-2}$.

5. $\dfrac{a^{\frac{2}{3}}b^{-\frac{4}{5}}}{x^{-3}y^{-5}} \times \dfrac{a^{-\frac{2}{3}}b^{-\frac{4}{5}}}{x^5y^3}$.

6. $\left(\dfrac{a^{-\frac{2}{5}}}{x^{\frac{1}{3}}y^{-\frac{2}{3}}}\right)^{-15}$.

7. $\left(\dfrac{x^{\frac{1}{4}}}{a^{-\frac{1}{3}}b^{-\frac{1}{2}}}\right)^{-12n}$.

8. $\left(\dfrac{x^{-\frac{2}{3}}-y^{-\frac{2}{3}}}{x^{-\frac{1}{3}}-y^{-\frac{1}{3}}}\right)^{-2}$.

Reduce to radicals with the same index:

9. $\sqrt{3}$ and $\sqrt[6]{4}$.

10. $\sqrt[8]{32}$ and $\sqrt[4]{5}$.

11. $\sqrt[3]{2}$ and $\sqrt[4]{3}$.

12. $\sqrt[3]{5}$ and $\sqrt[9]{25}$.

13. $\sqrt{5}$, $\sqrt[3]{2}$, and $\sqrt[4]{3}$.

14. $\sqrt{3}$, $\sqrt[5]{8}$, and $\sqrt[7]{4}$.

15. $\sqrt[3]{\frac{1}{2}}$, $\sqrt[4]{\frac{5}{8}}$, and $\sqrt{\frac{3}{5}}$.

16. $\sqrt[5]{\frac{2}{9}}$, $\sqrt{\frac{7}{10}}$, and $\sqrt[10]{\frac{121}{320}}$.

17. $\sqrt[7]{\frac{5}{7}}$, $\sqrt{\frac{3}{2}}$, and $\sqrt[14]{\frac{2}{11}}$.

18. $\sqrt[7]{0.3}$, $\sqrt[6]{\frac{2}{3}}$, and $\sqrt[21]{\frac{12}{17}}$.

19. $\sqrt{a}$, $\sqrt[3]{b}$, and $\sqrt[6]{c}$.

20. $\sqrt[7]{x^3}$ and $\sqrt[3]{x^5}$.

21. $\sqrt[3]{a^2}$, $\sqrt[4]{a^3}$, and $\sqrt[6]{a^5}$.

22. $\sqrt{\dfrac{x}{y^2}}$, $\sqrt[5]{\dfrac{y^3}{z}}$, and $\sqrt[3]{\dfrac{z}{y}}$.

23. $\sqrt[6]{\dfrac{m^2}{n^3}}$, $\sqrt[5]{\dfrac{1}{y^4}}$, and $\sqrt[3]{\dfrac{n}{y^2}}$.

24. $\sqrt[7]{\dfrac{x^2}{y}}$, $\sqrt{\dfrac{z}{y^2}}$, and $\sqrt[3]{\dfrac{4}{z}}$.

25. $\sqrt[m]{\dfrac{1}{x^2}}$, $\sqrt[n]{x}$, and $\sqrt[p]{\dfrac{1}{x}}$.

Combine by performing the indicated additions and subtractions, reducing to similar surds when necessary:

26. $2\sqrt{3}-5\sqrt{3}+9\sqrt{3}$.

27. $4\sqrt[3]{4}-3\sqrt[3]{4}+2\sqrt[3]{4}$.

28. $3\sqrt{2}+\sqrt{32}$.

29. $\sqrt{2}+3\sqrt{32}-\frac{1}{2}\sqrt{128}$.

30. $5\sqrt[3]{4}+2\sqrt[3]{32}-\sqrt[3]{108}$.

31. $\frac{1}{2}\sqrt[3]{5}+2\frac{1}{2}\sqrt[3]{5}+\frac{1}{4}\sqrt[3]{40}$.

32. $2\sqrt{175}-3\sqrt{63}+5\sqrt{28}$.

33. $3\sqrt{a}+b\sqrt{a}-\sqrt{a^3}$.

34. $\sqrt[3]{27c^4}-\sqrt[3]{8c^4}+\sqrt[6]{64c^8}$.

35. $\sqrt{a^2x}-\sqrt[4]{a^8x^6}+\sqrt{a^4x^3}$.

36. $\sqrt[3]{a^3b^3c}+\sqrt[6]{a^{12}b^{12}c^8}-ab\sqrt[3]{c^7}$.

Reduce to the form $\sqrt{x}+\sqrt{y}$:

37. $\sqrt{4+2\sqrt{3}}$.

38. $\sqrt{3+\sqrt{5}}$.

39. $\sqrt{2+\sqrt{3}}$.

40. $\sqrt{8+\sqrt{15}}$.

41. $\sqrt{5-\sqrt{21}}$.

42. $\sqrt{10+2\sqrt{21}}$.

43. $\sqrt{7+2\sqrt{10}}$.

44. $\sqrt{7-4\sqrt{3}}$.

45. $\sqrt{13-2\sqrt{30}}$.

46. $\sqrt{11-4\sqrt{7}}$.

Perform the following multiplications and divisions:

47. $(3 + 2\sqrt{2})(3 - 2\sqrt{2})$.

48. $(5 + 2\sqrt{3})(3 - 5\sqrt{3})$.

49. $(2\sqrt{6} - 3\sqrt{5})(\sqrt{3} + 2\sqrt{2})$.

50. $(\sqrt{7} - \sqrt{3})(\sqrt{2} + \sqrt{5})$.

51. $(\sqrt[3]{9} - 2\sqrt[3]{4})(4\sqrt[3]{3} + \sqrt[3]{2})$.

52. $(\sqrt{a+b} + \sqrt{a})(\sqrt{a+b} - \sqrt{a})$.

53. $\sqrt{m + \sqrt{n}}\,\sqrt{m - \sqrt{n}}$.

54. $\sqrt{a^2 - 1} \times \sqrt{\dfrac{a-1}{a+1}}$.

55. $\sqrt{a\sqrt[3]{a^2}} \times \sqrt{\sqrt[3]{a}}$.

56. $\sqrt[3]{x\sqrt{x^6}} \times \sqrt[3]{x\sqrt[3]{x^2}}$.

57. $\sqrt[3]{x^4y^2} \times \sqrt[5]{x^4y^4}$.

58. $\sqrt[4]{\dfrac{ab^2}{x^3}} \times \sqrt[6]{\dfrac{cx^5}{a^2b}}$.

59. $\sqrt{28} \div \sqrt{7}$.

60. $\sqrt{18} \div \sqrt{3}$.

61. $\sqrt{32} \div \sqrt{2}$.

62. $\sqrt[3]{56} \div \sqrt[3]{7}$.

63. $\sqrt[4]{243} \div \sqrt[4]{3}$.

64. $\sqrt[3]{12} \div \sqrt{6}$.

65. $\sqrt{54} \div \sqrt[4]{36}$.

66. $\sqrt{x^3} \div \sqrt[9]{x^8}$.

67. $\sqrt[5]{a^4} \div \sqrt[7]{2a^3}$.

68. $\sqrt[11]{a^5b^9} \div \sqrt[7]{a^4b^3}$.

69. $\sqrt[6]{27^{-1}z^9} \div \sqrt{6z^5}$

70. $\sqrt[4]{8x^3y^2} \div 2x^2y^3$.

Express with fractional exponents instead of radicals:

71. $(\sqrt[3]{m^2})^5$.

72. $(\sqrt{n^3})^4$.

73. $(\sqrt[4]{a^5})^5$.

74. $(\sqrt[8]{x^2y^3})^7$.

75. $(-\sqrt[3]{a^2})^4$.

76. $(-\sqrt{a^3})^5$.

77. $\left(\sqrt[3]{\sqrt{a^5}}\right)^{10}$.

78. $\left(\sqrt[5]{\sqrt[4]{16a^{17}}}\right)^{15}$.

79. $\left(\sqrt[4]{(x+y)^3}\right)^5$.

80. $\left(\sqrt[r]{\sqrt[s]{x^my^n}}\right)^t$.

Rationalize the denominator of:

81. $\dfrac{1}{\sqrt{2}}$.

82. $\dfrac{a}{\sqrt{b}}$.

83. $\dfrac{a}{\sqrt[3]{a^2}}$.

84. $\dfrac{a}{1 + \sqrt{a}}$.

85. $\dfrac{a}{1 - \sqrt{a}}$.

86. $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}$.

87. $\dfrac{3 + 2\sqrt{a^3}}{2\sqrt{a^3} - 3}$.

88. $\dfrac{1}{\sqrt{x+y} - \sqrt{x-y}}$.

89. $\dfrac{\sqrt{8} + \sqrt{7}}{\sqrt{7} - \sqrt{2}}$.

90. $\dfrac{\sqrt{13} - \sqrt{10}}{\sqrt{10} + \sqrt{13}}$.

91. $\dfrac{1}{a\sqrt{b} + c\sqrt{d}}$.

92. Calculate to three decimal places the values of the fractions in exercises 89 and 90.

Perform the following operations and simplify results:

93. $\sqrt[5]{a\sqrt[3]{a^2}} \times \sqrt[6]{\sqrt[4]{a^9}} \times \sqrt[3]{a^2\sqrt[8]{a^7}}$.

94. $(x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})(x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})$.

95. $(2a^{-\frac{5}{4}} - 3a^{-\frac{3}{4}} + a^{-\frac{1}{4}} - 2)(a^{-\frac{3}{4}} - 2a^{-\frac{1}{4}} + 3)$.

96. Write out the result of replacing $a^{-\frac{1}{4}}$ by b in exercise 95.

97. $\left(a^{\frac{3}{x}} - 2a^{\frac{2}{x}} + 3a^{\frac{1}{x}}\right)\left(2a^{\frac{2}{x}} - a^{\frac{1}{x}} + 2\right)$.

98. $\left(y^{\frac{3}{n}} - ay^{\frac{2}{n}} + 3by^{\frac{1}{n}} - c\right)\left(y^{\frac{2}{n}} + by^{\frac{1}{n}} - cy^0\right)$.

99. $(2a^{-1}b^{-\frac{2}{3}} - 3a^{-\frac{3}{5}}b^{-\frac{4}{5}})^2$.

100. $(a^{-\frac{1}{2}} + b^{-\frac{1}{3}})^3$.

101. $(x^{\frac{1}{2}} - y^{\frac{2}{3}})^3$.

102. $(1^{-2} - n^{-\frac{2}{3}})^4$.

103. $(m^{-2} + m^{-\frac{1}{2}})^4$.

104. $(a^{-1}x^{-2} - ax^2)^4$.

105. $(a^{\frac{2}{3}} + a^{\frac{3}{4}} - a^{\frac{4}{5}})^2$.

106. $(2a^{\frac{2}{3}} - 3b^{\frac{2}{5}} - 4c^{\frac{1}{2}})^2$.

107. $(a^{\frac{1}{4}} - 2b^{\frac{1}{3}} + 3c^{\frac{1}{2}} - 4d^{\frac{1}{5}})^2$.

108. $(x^{\frac{4}{5}}y^{\frac{1}{3}} - 2x^{\frac{3}{5}}y^{\frac{2}{3}} + 3x^{\frac{2}{5}}y - 2x^{\frac{1}{5}}y^{\frac{4}{3}})^2$.

109. Write out the result of replacing $x^{\frac{1}{5}}$ by u and $y^{\frac{1}{3}}$ by v in exercise 108.

110. $(x - 1) \div (\sqrt[3]{x} - 1)$.

111. $(x + 1) \div (\sqrt[5]{x} + 1)$.

112. $(\sqrt{x} - \sqrt{y}) \div (\sqrt[4]{x} - \sqrt[4]{y})$.

113. $(a^{\frac{1}{6}} - b^{\frac{1}{6}}) \div (a^{\frac{1}{18}} - b^{\frac{1}{18}})$.

114. $(x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}) \div (\sqrt{x} - \sqrt{y})$.

115. $(a^{\frac{5}{2}} - a^2 - 4a^{\frac{3}{2}} + 6a - 2\sqrt{a}) \div (a^{\frac{3}{2}} - 4\sqrt{a} + 2)$.

116. $(a^{\frac{1}{2}} - b^{\frac{1}{2}} - c^{\frac{1}{2}} + 2\sqrt[4]{bc})(a^{\frac{1}{4}} + b^{\frac{1}{4}} - c^{\frac{1}{4}})$.

Express the following in the form $a\sqrt{-1}$.

117. $\sqrt{-9}$; $\sqrt{-25}$; $\sqrt{-81}$.

118. $\sqrt{-a^2}$; $\sqrt{-b^2}$; $\sqrt{-x^{2n}}$.

119. $\sqrt[4]{-81}$.

120. $\sqrt[6]{-64}$.

121. $\sqrt[8]{-256}$.

122. $\sqrt[10]{-a^{20}}$.

123. $\sqrt{-25} - \sqrt{-49} + \sqrt{-121}$.

124. $\sqrt{-a^4} + \sqrt{-a^2} - \sqrt{-4a^4}$.

125. $\sqrt{-(m+n)^2} + \sqrt{-(m-n)^2} - \sqrt{-m^2}$.

Multiply and reduce to the form $a + b\sqrt{-1}$: $(i \equiv \sqrt{-1})$.

126. $(a + b\sqrt{-1})(a - b\sqrt{-1})$.

127. $(3 + 5i)(4 - 7i)$.

128. $(x + 2i)(y - 3i)$.

129. $(\sqrt{8} + i\sqrt{12})(\sqrt{2} + i\sqrt{3})$.

130. $(-1 + i\sqrt{3})^3$.

131. $\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^4$.

Reduce to the form $a + bi$ by rationalizing the denominator:

132. $\frac{2}{3 + 2i}$

133. $\frac{1 + i}{1 - i}$.

134. $\frac{a + bi}{a - bi}$.

135. $\frac{a + i\sqrt{x}}{a - i\sqrt{x}}$.

136. $\frac{36}{7 + 2\sqrt{-5}}$.

137. $\frac{1 - i^3}{(1 - i)^3}$.

Clear the following equations of radicals:

(*Example.* To clear the equation $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ of radicals put

$$\sqrt{x} + \sqrt{y} = 1 - \sqrt{z};$$

squaring, $$x + y + 2\sqrt{xy} = 1 + z - 2\sqrt{z},$$

or, $$x + y - z - 1 = -2\sqrt{xy} - 2\sqrt{z}.$$

Squaring again, $$(x + y - z - 1)^2 = 4xy + 4z + 8\sqrt{xyz},$$

or $$(x + y - z - 1)^2 - 4(xy + z) = 8\sqrt{xyz}.$$

Squaring again, $[(x + y - z - 1)^2 - 4(xy + z)]^2 = 64xyz$. q.e.f.

138. $\sqrt{x + 4} = 4$.

139. $\sqrt{2x + 6} = 3$.

140. $\sqrt[3]{x + 1} = 2$.

141. $\sqrt[4]{ax + b} = c$.

142. $\sqrt{x} + \sqrt{y} = 1$.

143. $\sqrt{x + 1} - \sqrt{x - 1} = 2$.

144. $\sqrt{32 + x} = 16 - \sqrt{x}$.

145. $\sqrt{x + 20} - \sqrt{x - 1} - 3 = 0$.

146. $\sqrt{x} - \frac{5}{\sqrt{x}} = \sqrt{x - 9}$.

147. $\sqrt{15 + \sqrt{2x + 80}} = 5$.

148. $\sqrt{3 + \sqrt{x}} = \sqrt{12 - 2\sqrt{x}}$.

149. $\sqrt{3 + \sqrt{x}} = \sqrt{11 - 3\sqrt{x}}$.

150. $\frac{\sqrt{8x + 1} + \sqrt{8x}}{\sqrt{8x + 1} - \sqrt{8x}} = 13$.

CHAPTER III

Logarithms. Binomial Theorem for Positive Integral Exponents

34. Logarithm. — The simple laws of operation for exponents have given rise to a method of calculation involving the use of a function called the *logarithm*. We shall first illustrate this method.

Suppose that we know the powers of 10 which are required to produce a set of numbers, as in the adjacent table, where the exponents are given to the nearest figure in the third decimal. *The exponent of* 10 *in each equation is called the common logarithm* (*or the logarithm to the base* 10) *of the number on the left.* Thus, the logarithm of 5.00 is 0.699, of 5.50 is 0.740, and so on. As equations, we write

$$\log_{10} 5.00 = 0.699,$$
$$\log_{10} 5.50 = 0.740,$$

and so on.

TABLE.

$$5.00 = 10^{0.699}$$
$$5.50 = 10^{0.740}$$
$$6.00 = 10^{0.778}$$
$$6.50 = 10^{0.813}$$
$$7.00 = 10^{0.845}$$
$$7.50 = 10^{0.875}$$
$$8.00 = 10^{0.903}$$
$$8.50 = 10^{0.929}$$
$$9.00 = 10^{0.954}$$
$$9.50 = 10^{0.978}$$
$$10.00 = 10^{1.000}$$

35. By aid of such a table *products* of numbers (within certain limits) can be obtained by *adding* the logarithms of the factors; also, *division* is reduced to *subtraction* of logarithms.

Example 1. Find the value of $6.5 \times 8.5 \times 9.5$.

We have
$$6.5 \times 8.5 \times 9.5 = 10^{0.813} \times 10^{0.929} \times 10^{0.978}$$
$$= 10^{0.813+0.929+0.978}$$
$$= 10^{2.720} = 10^2 \times 10^{0.720}.$$

Now 0.720 lies almost exactly midway between 0.699 and 0.740; hence the number corresponding to $10^{0.720}$ will be midway between 5.00 and 5.50 and is equal to 5.25. (This involves the assumption that a logarithm changes proportionately to the change in the number, an assumption which is not exactly, but very nearly, true except for numbers near zero, provided the changes in the numbers are small.)

Therefore,
$$6.5 \times 8.5 \times 9.5 = 10^2 \times 10^{0.720} = 100 \times 5.25 = 525.$$

The exact value is 524.875.

Definition. **Interpolation** is the process of calculating numbers intermediate to those given in a table.

Example 2. — Find the value of $\frac{6.25 \times 7.20}{5.75}$.

Let $10^a = 6.25;\ 10^b = 7.20;\ 10^c = 5.75.$

Then
$$\frac{6.25 \times 7.20}{5.75} = \frac{10^a \times 10^b}{10^c} = 10^{a+b-c}.$$

Since 6.25 lies halfway between 6.00 and 6.50, we take for a the value halfway between the corresponding exponents, so that $a = 0.795$ (more exactly 0.7955). To get b, we note that 7.20 lies $\frac{2}{5}$ of the way from 7.00 to 7.50; hence we take for b the number lying in the corresponding position between the exponents 0.845 and 0.875. Therefore

$$b = 0.845 + \tfrac{2}{5} \times 0.030 = 0.857.$$

Similarly, $c = 0.759.$

Hence,
$$\frac{6.25 \times 7.20}{5.75} = 10^{0.795+0.857-0.759} = 10^{0.893}.$$

The corresponding number lies between 7.50 and 8.00, and nearer the latter. Since our exponent, 0.893, lies $\frac{18}{28}$ of the way from 0.875 to 0.903, we find the number lying in the corresponding position between 7.50 and 8.00, that is,

$$7.50 + \tfrac{18}{28} \times 0.50 = 7.50 + 0.32 = 7.82.$$

Therefore,
$$\frac{6.25 \times 7.20}{8.50} = 7.82 \text{ approximately}.$$

This result is correct to two decimals.

36. By the aid of our table, powers and roots of numbers may be found by applying the operations of multiplication and division, respectively to their logarithms.

Example. Find the value of $\sqrt[4]{9.35^3}$.

We have $\sqrt[4]{9.35}^{\,3} = (9.35)^{\frac{3}{4}}.$

Let $9.35 = 10^a;$ then $(9.35)^{\frac{3}{4}} = 10^{\frac{3}{4}a}.$

From the table, $a = 0.954 + \frac{7}{10} \times 0.024 = 0.971.$

Therefore, $\sqrt[4]{9.35^3} = 10^{0.728} = 5.00 + \frac{22}{41} \times 0.50 = 5.35.$

A more accurate value is 5.335, so that the second decimal of our result is slightly in error.

Obviously the calculation of the last result by the methods of arithmetic would be very tedious, and with a slight increase in the complexity of the exponent these methods would become quite useless.

We shall now consider the general theory of the method illustrated above.

37. Logarithm of a Number. — Let a be a certain fixed number, n any other number, and let x be the exponent of a required to produce n. *Then x is the logarithm of n to the base a.*

As equations,

if $a^x = n$, *then* $x = \log_a n$.

We give below some very simple *tables of logarithms.*

Number.	Logarithm Base = 2.	n.	$\log_{10} n$.	n.	$\log_{10} n$.
$\frac{1}{8}$	− 3	.001	− 3	5.00	0.699
$\frac{1}{4}$	− 2	.01	− 2	5.50	0.740
$\frac{1}{2}$	− 1	.1	− 1	6.00	0.778
1	0	1.0	0	6.50	0.813
2	1	10	1	7.00	0.845
4	2	100	2	7.50	0.875
8	3	1000	3	8.00	0.903

38. Exercises.

1. What is the value of $\log_a 1$?

2. What are the logarithms of 8, 16, 64, 128 to the base 2?

3. What are the logarithms of 8, 16, 64, 128 to the base $\frac{1}{2}$?

4. What are the logarithms of $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{243}$, to the base 3? to the base $\frac{1}{3}$?

5. What are the logarithms of $\frac{125}{8}$ and $\frac{625}{16}$ to the base $\frac{2}{5}$?

6. What are the logarithms of 2, 4, 8 to the base 64?

7. What is the base, if $\log 2 = 1$? if $\log a = 1$?

8. What is the base, if $\log \frac{1}{3} = 4$? if $\log 25 = -2$?

9. What is the base if $\log 49 = 2$? if $\log .0081 = 4$?

10. What is $\log_2 (-4)$?

11. Why would it be inconvenient to use a negative number as the base of a system of logarithms?

12. If $n = (e^{x+y})^{x-y}$, find $\log_e n$.

13. If $x = \sqrt[p]{e}\,\sqrt[q]{e}\,(e^{p+q})$, find $\log_e x$.

14. If $a = \left[(10^{x^3-y^3})^{\frac{1}{x-y}}\right]^{x^2-xy+y^2}$, find $\log_{10} a$.

15. Show that $a^{\log_a x} = x$.

39. Laws of Operation with Logarithms. — Since a *logarithm is an exponent,* the laws of operation for logarithms are the same as those for exponents.

Let x be the logarithm of m, y that of n, the base being a. Then

$$\begin{cases}\log_a m = x, \\ \log_a n = y,\end{cases} \quad \text{or} \quad \begin{cases} a^x = m, \\ a^y = n.\end{cases}$$

Hence

$$mn = a^{x+y} \quad \text{and} \quad \frac{m}{n} = a^{x-y},$$

or,
$$\log_a mn = x + y = \log_a m + \log_a n,$$

and
$$\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$$

We have therefore the rules:

I. *The logarithm of a product equals the sum of the logarithms of the factors.*

II. *The logarithm of a fraction equals the logarithm of the numerator minus the logarithm of the denominator.*

Also, if as before,

$$\log_a m = x, \quad \text{so that} \quad m = a^x,$$

then, if p and q be any real numbers,

$$m^p = a^{px} \quad \text{and} \quad \sqrt[q]{m} = a^{\frac{x}{q}}.$$

Hence,
$$\log_a m^p = px = p \log_a m,$$

and
$$\log_a \sqrt[q]{m} = \frac{x}{q} = \frac{1}{q} \log_a m.$$

We have therefore two additional rules:

III. *The logarithm of any power of a number equals the exponent of the power times the logarithm of the number.*

IV. *The logarithm of any root of a number equals the logarithm of the number divided by the index of the root.*

(Rule III contains rule IV, since the power in question may be fractional.)

40. The following facts regarding logarithms should also be carefully noted.

(a) In any system the logarithm of the base is 1.

For $\quad a^1 = a. \quad \therefore \log_a a = 1.$

(b) In any system the logarithm of 1 is 0.

For $\quad a^0 = 1. \quad \therefore \log_a 1 = 0.$

(*c*) In any system whose base is greater than unity, the logarithm of 0 is $-\infty$.

For if $a^x = m$ and $a > 1$, then if x is a large negative number, m will be small. As x increases indefinitely, always being negative, m approaches zero. That is,

$$a^{-\infty} = 0 \text{ if } a > 1; \qquad \therefore \log 0 = -\infty.$$

(*d*) A negative number has no (real) logarithm, the base being positive.

(*e*) As a number varies from 0 to $+\infty$, its logarithm varies from $-\infty$ to $+\infty$, the base being greater than 1.

When the number is greater than 1, its logarithm is positive.

When the number is less than 1, its logarithm is negative.

41. Exercises. (See Appendix for tables and explanation of their use.)

1. Discuss (*c*) of (**40**) when the base is less than unity.
2. Discuss (*e*) of (**40**) when the base is less than unity.

In the following exercises, the base is understood to be 10, and four-place logarithms are to be used.

3. Find log 831, log 8.31, log .831, and log .0831.
4. Find log 78.03, log .073, log .00284.
5. Find the approximate value of $564.1 \times .0065$.
6. Calculate $\sqrt[3]{154.2}$ and $(7.541)^3$.
7. Calculate $518 \div 313$ and $25.03 \div 2.14$.
8. Calculate $.001022 \div .0000513 \times 1.415$.
9. Calculate $17\sqrt[3]{29}$ and $41\sqrt{0.512}$.
10. Calculate $\sqrt[6]{0.35^4} \times \sqrt[3]{0.47^2}$.
11. Calculate $\dfrac{(.00165)^3(.0764)^2}{(.00346)^4}$.
12. Calculate $\sqrt[3]{214 - \sqrt{214}}$.

Write as a single term:

13. $\log a - \log b + \log c - \log d$.
14. $3 \log x - 4 \log y + 2 \log z$.
15. $\frac{1}{2} \log u + \frac{1}{3} \log v - \frac{1}{4} \log w$.
16. $\log \frac{a}{b} + \log \frac{b}{c} + \log \frac{c}{d} - \log \frac{ax}{dy}$.
17. $3 \log a - \log (x + y) - \frac{1}{3} \log (ax + b) + \log \sqrt{ax + b}$.

The Binomial Theorem for Positive Integral Exponents

42. This theorem is used to express $(a + b)^n$ in expanded form. We shall here obtain the formula assuming n to be a positive integer; the proof for other values of n will be found in (**221**).

By actual multiplication we have

$$(a + b)^2 = a^2 + 2\,ab + b^2,$$
$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3,$$
$$(a + b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$$

Here we observe the following laws:

I. The *number of terms* is 1 greater than the exponent of the binomial.

II. The *exponent of* a in the first term equals that of the binomial and decreases by unity in each succeeding term. *The exponent of* b is 1 in the second term and increases by unity in each succeeding term.

III. The *coefficient* of the first term is 1, and of the second term the exponent of the binomial. If the coefficient of any term be multiplied by the exponent of a in that term, and the result be divided by the exponent of b plus 1, we obtain the coefficient of the next following term.

43. Now let

$$(1)\quad (a+b)^n = a^n + c_1a^{n-1}b + c_2a^{n-2}b^2 + \cdots + c_{m-1}a^{n-(m-1)}b^{m-1} + c_ma^{n-m}b^m + c_{m+1}a^{n-(m+1)}b^{m+1} + \cdots.$$

We have here assumed laws I and II and have written the exponents accordingly. Assuming also law III, we shall have

$$(2)\quad c_1 = n;\ c_2 = \frac{n-1}{2}c_1;\ c_m = \frac{n-(m-1)}{m}c_{m-1};\ c_{m+1} = \frac{n-m}{m+1}c_m.$$

We can now show that the same laws are true for the expansion of $(a + b)^{n+1}$.

Multiplying (1) by $(a + b)$ and collecting like terms we have

$$(3)\quad (a+b)^{n+1} = a^{n+1} + (1+c_1)a^{(n+1)-1}b + (c_1+c_2)a^{(n+1)-2}b^2 + \cdots + (c_{m-1}+c_m)a^{n+1-m}b^m + (c_m+c_{m+1})a^{(n+1)-(m+1)}b^{m+1} + \cdots$$

The number of terms will be $n + 2$, since the exponent of a starts with $n + 1$ and decreases to 0. Hence law I is still true. Also law II is evidently true.

According to the third law, we should have

$$(1 + c_1) = n + 1;\ c_1 + c_2 = \frac{(n+1) - 1}{2}(1 + c_1);\ \ldots$$

$$(c_m + c_{m+1}) = \frac{(n+1) - m}{m+1}(c_{m-1} + c_m).$$

These equations all become identities on substituting from (2).

Therefore all three laws are true for the expansion of $(a + b)^{n+1}$ *provided* that they are true for the expansion of $(a + b)^n$. But they are true for $(a + b)^4$, hence for $(a + b)^5$, hence for $(a + b)^6$, and so on, for any positive integral exponent.

This method of proof is called *proof by induction.*

Writing out the values of several coefficients we have,

$$c_1 = n;\ c_2 = \frac{n(n-1)}{1 \cdot 2};\ c_3 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3};\ \ldots$$

$$c_m = \frac{n(n-1)(n-2)\ \ldots\ (n-m+1)}{1 \cdot 2 \cdot 3 \cdot\ \ldots\ m},$$

where c_m is the coefficient of the $(m + 1)$th term.

In place of $1 \cdot 2 \cdot 3 \cdot \ldots m$ we use the symbol $\underline{|m}$ or $m!$ (in either case, read "factorial m"). Then equation (1) becomes

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{\underline{|2}}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}b^3$$

$$+ \cdots + \frac{n(n-1)(n-2)\ \cdots\ (n-m+1)}{\underline{|m}}a^{n-m}b^m + \cdots.$$

When $a = 1$ and $b = x$ we have,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots.$$

44. The expansion of $(a + b)^n$ may be reduced to that of $(1 + x)^n$ thus:

$$(a + b)^n = a^n\left(1 + \frac{b}{a}\right)^n = a^n\left[1 + n\frac{b}{a} + \cdots\right].$$

In place of c_m to denote the coefficient of the $(m + 1)$th term of the expansion of $(a + b)$, the symbols ${}_nc_m$ or $\binom{n}{m}$ are often used. These are called the *binomial coefficients.*

Table of Binomial Coefficients

$n = 0$	1
$n = 1$	1 1
$n = 2$	1 2 1
$n = 3$	1 3 3 1
$n = 4$	1 4 6 4 1
$n = 5$	1 5 10 10 5 1

Example 1.
Expand $(a^{\frac{1}{2}} - 2b^2)^4$.

$$[(a^{\frac{1}{2}}) + (-2b^2)]^4 = (a^{\frac{1}{2}})^4 + 4(a^{\frac{1}{2}})^3(-2b^2) + 6(a^{\frac{1}{2}})^2(-2b^2)^2 + 4(a^{\frac{1}{2}})(-2b^2)^3 + (-2b^2)^4$$
$$= a^2 - 8a^{\frac{3}{2}}b^2 + 24ab^4 - 32a^{\frac{1}{2}}b^6 + 16b^8.$$

Example 2.
Find the fifth term in the expansion of $(x^{-\frac{2}{3}} - \frac{1}{2}y^3)^8$.
This term will be

$$\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}(x^{-\frac{2}{3}})^4\left(-\frac{1}{2}y^3\right)^4 = \frac{35}{8}x^{-\frac{8}{3}}y^{12}.$$

45. Exercises. Expand:

1. $(x-y)^5$.
2. $(2a-3b)^6$.
3. $(a^{-1}+b^{-2})^4$.
4. $(x^{-\frac{1}{2}} - y^{\frac{1}{2}})^7$.
5. $(x^{\frac{2}{3}} + y^{\frac{1}{4}})^6$.
6. $(2p^2 - 3q^{\frac{1}{2}})^5$.
7. $(ax+by)^8$.
8. $(\frac{1}{2}u^{-2} + 2v^2)^7$.
9. $(\sqrt{2x} - \sqrt[3]{3y})^6$.
10. $\left(\frac{1}{2a} + \frac{3}{b}\right)^5$.
11. $\left(\sqrt{5}\,m^{-1} + \frac{n^{-2}}{\sqrt{5}}\right)^4$.
12. $(1+a^x)^7$.
13. $(a^x+b^y)^6$.
14. $(a^{x+y} + a^{x-y})^5$.
15. $(x^{n^2} - y^{m^2})^6$.
16. $(x^x - y^y)^4$.
17. $(\sqrt[a]{x} + \sqrt[b]{y})^8$.
18. $(x^{\sqrt{a}} - a^{\sqrt{x}})^7$.
19. $(e^{2x} + xe^{-2x})^5$.
20. $\left(\frac{m^2}{n^2} - \frac{n^2}{m^2}\right)^6$.

To expand a trinomial or other polynomial, proceed by grouping the terms in two groups, thus:

$$(x+y+z)^3 = [x+(y+z)]^3$$
$$= x^3 + 3x^2(y+z) + 3x(y+z)^2 + (y+z)^3.$$

The expansion may now be completed by the formula.

21. $(x+y-z)^3$.
22. $(\sqrt{x} - \sqrt{y} + \sqrt{z})^3$.
23. $(1+2a+3a^2)^4$.
24. $(x-y+u-v)^3$.
25. $(1+2x+3x^2+4x^3)^3$.

Calculate:

26. the 6th term of $(3 + 2x^2)^9$.

27. the 5th term of $(\sqrt{2c} + \sqrt{3d})^{10}$.

28. the 8th term of $(2b^{\frac{3}{2}} - \frac{1}{2}\sqrt{x})^8$.

29. the 12th term of $(3y^{\frac{3}{2}} + \frac{1}{3}y^{\frac{2}{3}})^{15}$.

30. the 10th term of $(\sqrt{\frac{2}{3}a^3} - \sqrt{\frac{1}{2}a^5})^{20}$.

46. Approximate Computation by Use of the Binomial Theorem. — When x is a small fraction, the terms of the formula

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \cdots$$

rapidly decrease. *In any numerical problem in which only approximate results are required, retain only enough terms of the expansion to obtain the desired degree of accuracy.*

It will often be found sufficient to use the simple formula,

$$(1 + x)^n = 1 + nx, \text{ approximately.}$$

Example 1. Calculate $(0.997)^4$ to three decimals.

$$(0.997)^4 = (1 - .003)^4 = 1 - 4 \times .003 = 0.988.$$

Exercise. Show that the terms neglected will not affect the third decimal place.

Example 2. Calculate $(2.05)^3$ to three decimals.

$$(2.05)^3 = 2^3(1 + .025)^3 = 8(1 + 3 \times .025 + 3 \times .000625 + \cdots)$$
$$= 8 \times 1.0769 = 8.615.$$

Exercises. Calculate to three decimal places the value of:

1. $(0.995)^5$. **2.** $(1.05)^7$. **3.** $(3\frac{1}{15})^4$.

4. $(2\frac{14}{15})^4$. **5.** $(3.998)^6$. **6.** $(8.0125)^2$.

7. Calculate the value of $(.99995)^7$ to seven decimals.

CHAPTER IV

Linear Equations

47. If $$X = Y, \quad \text{and} \quad m = n,$$

then $$X + m = Y + n, \quad X - m = Y - n,$$

$$mX = nY, \quad \text{and} \quad \frac{1}{m}X = \frac{1}{n}Y.$$

That is, *if both members of an equation be increased or diminished, multiplied or divided, by the same or equal quantities, the results are equal.*

Also if $$X = Y, \text{ then } X^n = Y^n,$$

n being an integer; that is, *if both members of an equation be raised to the same integral power, positive or negative, the results are equal.*

If $$X = Y, \text{ then } \sqrt[n]{X} = \sqrt[n]{Y},$$

provided the corresponding nth roots of X and Y are selected.

If $$X + m = Y,$$

then subtracting m from both members,

$$X = Y - m.$$

That is, *a term may be transposed from one side of an equation to the other provided its sign is changed at the same time.*

When the members of an equation involve sums or differences of fractions, *the equation may be cleared of fractions by multiplying both members by the L. C. D. of the several fractions.*

48. Linear Equation. — If x be an unknown quantity related to the known quantities a and b through the equality $ax + b = 0$, this equation being called *the standard form of the linear equation in one unknown,* we obtain the value of x as

$$x = -\frac{b}{a}.$$

Every linear equation in one unknown may be solved by reducing it to standard form and applying the last formula.

The reduction of an equation to standard form will involve some or all of the following steps:

1. Clearing of radicals. (**33**, after exercise 137.)
2. Clearing of fractions.
3. Expanding products or powers of polynomials.
4. Transposing and cancelling.
5. Collecting terms.

To verify the value found, substitute it in the given equation. The result should be an identity.

49. *Example* 1. Solve for x: $(1 + b)\,x + ab = b\,(a + x) + a.$

Expanding the products: $x + \cancel{bx} + \cancel{ab} = \cancel{ab} + \cancel{bx} + a.$

Cancelling like terms: $x = a$

Check: $(1 + b)\,a + ab = b\,(a + a) + a.$

Example 2. $$\frac{1}{2} + \frac{2}{x+2} = \frac{x+2}{2\,x}.$$

Multiplying by the *L. C. D.*, $2\,x\,(x + 2)$:

$$x\,(x + 2) + 4\,x = (x + 2)^2.$$

Expanding: $x^2 + 2\,x + 4\,x = x^2 + 4\,x + 4.$

Cancelling: $2\,x = 4$ or $x = 2.$

Check: $\frac{1}{2} + \frac{2}{4} = \frac{4}{4}.$

Example 3. Solve for x: $\sqrt{x + 20} - \sqrt{x - 1} - 3 = 0.$

Transposing: $\sqrt{x + 20} = \sqrt{x - 1} + 3.$

Squaring: $x + 20 = x - 1 + 6\sqrt{x - 1} + 9,$

or, $2 = \sqrt{x - 1}.$

Squaring: $4 = x - 1$ or $x = 5.$

Check: $\sqrt{25} - \sqrt{4} - 3 = 0.$

50. Infinite Solutions. — Consider the equation $\dfrac{1}{x+1} = \dfrac{1}{x-1}.$ Since $x + 1$ cannot equal $x - 1$ for any value of x, there is no value of x which will satisfy the given equation.

But if we substitute in the given equation successively $x = 10$, 100, 1000, etc., the equation is more nearly satisfied, the larger the value of x. We can take x so large as to make the difference between the two members of the equation as small as we please; for this difference is

$$\frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{x^2-1}.$$

For brevity we say that $x = \infty$ *is a solution of the equation*, meaning thereby that as increasing values of x are substituted, the equation is more and more nearly satisfied.

Substituting formally $x = \infty$, we obtain

$$\frac{1}{\infty+1} = \frac{1}{\infty-1} \quad \text{or} \quad 0 = 0.$$

The equation of example 2 of (**49**) admits the solution $x = \infty$. This will be evident on putting ∞ for x in

$$\frac{1}{2} + \frac{2}{x+2} = \frac{x+2}{2\,x} = \frac{1}{2} + \frac{1}{x}.$$

51. Exercises. Solve for x, including infinite solutions when present:

1. $5\,(a-x) = 3\,(b-x)$.
2. $p\,(x-1) + x = q - p$.
3. $a\,(bx - c) = ac - abx$.
4. $\dfrac{m-x}{n} = \dfrac{x-n}{m}$.
5. $\dfrac{m}{x} + n = \dfrac{p}{x} + q$.
6. $\dfrac{a+bx}{c+dx} = \dfrac{a}{c}$.
7. $\dfrac{a+bx}{c+dx} = \dfrac{b}{d}$.
8. $\dfrac{a+b}{a+bx} = \dfrac{c-d}{c-dx}$.
9. $\dfrac{\dfrac{a+1}{x}}{\dfrac{b}{x}} = \dfrac{a+x}{b+x}$.
10. $\dfrac{\dfrac{cx+d}{m}}{\dfrac{cx}{d}} = \dfrac{2\,d}{m}$.
11. $\dfrac{m}{x-m} - \dfrac{n}{x-n} = \dfrac{m-n}{x-a}$.
12. $\sqrt{x+15} + \sqrt{x-13} = 14$.
13. $3\sqrt{16\,x+9} + 9 = 12\sqrt{4\,x}$.
14. $\sqrt{x} - \sqrt{x-5} - \sqrt{5} = 0$.
15. $\sqrt{\frac{15}{4}+x} = \frac{3}{2} + \sqrt{x}$.
16. $\dfrac{1}{x} + \dfrac{2}{x} + \dfrac{3}{x} = 0$.

52. Graphic Solution of Linear Equations. — Suppose that a given equation has been reduced to the standard form,

$$ax + b = 0.$$

The solution of the equation is that value of x which reduces the binomial to 0. For brevity, let us represent the binomial by y, so that

$$y = ax + b.$$

Then we want that value of x for which $y = 0$. If now we form a table which gives the values of y corresponding to a series of assumed values of x, we may obtain from it by inspection the exact or approximate value of x for which y is zero.

Example.

Let $2x - 1 = 0$ so that $y = 2x - 1$.

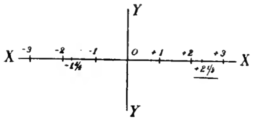
Corresponding values of x and y are:

$$x = -2, \quad -1, \quad 0, \quad +1, \quad +2, \quad +3, \ldots .$$
$$y = -5, \quad -3, \quad -1, \quad +1, \quad +3, \quad +5, \ldots .$$

By inspection we see that $y = 0$ when x lies between 0 and 1.

53. Graph of the Equation $y = 2x - 1$. — We shall now represent the corresponding values of x and y graphically.

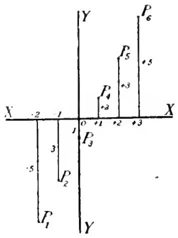
Divide the plane of the paper into four quarters or **quadrants** by drawing two mutually perpendicular lines, XX and YY, intersecting at O. (See figure.) Adopting any convenient unit of length (say one-fourth of an inch, or one side of a square of the cross-section paper), mark on XX a series of points whose distances from O shall equal the assumed values of x. When x is positive, the distance is laid off to the right from O; when x is negative, to the left.



In this way all positive and negative integral values of x are represented by a series of segments having a common starting point O, and ending in a series of equally spaced points on the line XX, each of which represents an integral value of x. Non-integral values of x are represented by segments whose end points fall between two points representing integral values. Thus in the figure are marked the points corresponding to $x = \pm 1$, ± 2, ± 3, $+2\frac{1}{3}$ and $-1\frac{3}{4}$.

Now to represent the value of y corresponding to a given value of x, mark the representative point of x on XX, and at this point lay off a segment perpendicular to XX and having a length equal to the value of y; this segment is drawn upward when y is positive, and downward when y is negative.

When we construct in this way the pairs of values of x and y given in the example of (**52**), we obtain the figure below. We thus get a series of points, P_1, P_2, . . . , P_6, whose distances from the line XX are the values of the binomial $2x - 1$ for the assumed values of x. Inspection of the figure shows that as x increases from -2 to $+3$, y (i.e. $2x - 1$) increases from -5 to $+5$; also that $y = 0$ between $x = 0$ and 1.



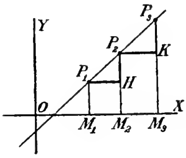
Exercise. By similar triangles, show that any three, and hence all the points marked in the figure, lie on a straight line.

By drawing a smooth curve (in this case a straight line) through a sufficient number of points P_1, P_2, . . . we obtain the *graph of the equation* $y = 2x - 1$. The points P_1, P_2 . . . are said to lie on this graph.

54. Graph of $y = ax + b$.—The graph of the equation $y = ax + b$ is a geometric picture which indicates the value of y corresponding to any assumed value of x.

We shall now show that *this graph is a straight line*, by showing that any three of its points are collinear.

Let x_1, x_2, and x_3 be any three values of x; let y_1, y_2, and y_3 be the corresponding values of y. Lay off the corresponding values (x_1, y_1), (x_2, y_2), and (x_3, y_3) so that (see figure)



$$x_1 = OM_1, \qquad y_1 = M_1P_1,$$
$$x_2 = OM_2, \qquad y_2 = M_2P_2,$$
$$x_3 = OM_3, \qquad y_3 = M_3P_3.$$

But since y_1 is the value of y obtained by putting $x = x_1$ in $y = ax + b$, and similarly for y_2 and y_3, we have

$$\begin{aligned} y_1 &= ax_1 + b \\ y_2 &= ax_2 + b \\ y_3 &= ax_3 + b \end{aligned} \quad \therefore \quad \begin{aligned} y_2 - y_1 &= a(x_2 - x_1), \\ y_3 - y_2 &= a(x_3 - x_2). \end{aligned}$$

Therefore,
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} \quad (= a).$$

But $y_2 - y_1 = M_2P_2 - M_1P_1 = M_2P_2 - M_2H = HP_2;$
$y_3 - y_2 = M_3P_3 - M_2P_2 = KP_3;$
$x_2 - x_1 = OM_2 - OM_1 = M_1M_2 = P_1H;$
and $x_3 - x_2 = OM_3 - OM_2 = M_2M_3 = P_2K.$

Substituting these in the two fractions above we obtain

$$\frac{HP_2}{P_1H} = \frac{KP_3}{P_2K}.$$

Therefore $\triangle\, P_1HP_2$ is similar to $\triangle\, P_2KP_3$.
Hence the points P_1, P_2, P_3 lie on a straight line.

Theorem: The graph of the equation $y = ax + b$ is a straight line.

Corollary: To construct the graph of the equation $y = ax + b$, construct two points on it and draw a straight line through them.

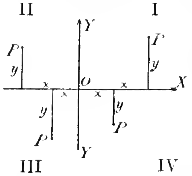
55. Exercises. Draw the graphs of the equations (each set to the same reference lines):

1. $y = x + 1$, $2y = 2x + 2$, $5x = 5x + 5$, $\frac{1}{2}y = \frac{1}{2}x + \frac{1}{2}$.
2. $y = 3x - 4$, $2y = 6x - 8$, $ky = 3kx - 4k$.
3. $y = x + 1$, $y = x + 2$, $y = x + 3$, $y = x - 1$.
4. $y = 3x - 4$, $y = 3x - 2$, $y = 3x$, $y = 3x + 1$.
5. $y = x + 1$, $y = 2x + 1$, $y = 3x + 1$, $y = \frac{1}{2}x + 1$.
6. $y = 3x - 4$, $y = 6x - 4$, $y = 9x - 4$, $y = \frac{3}{2}x - 4$.
7. $y = -x + 1$, $y = -3x - 4$.
8. $y = x - 1$, $y = 3x + 4$.

Explain the effect on the graph of $y = ax + b$, of:

9. Multiplying the equation through by a constant.
10. Changing the value of b.
11. Changing the value of a.
12. Changing the sign of b.
13. Changing the sign of a.

56. Coördinates. — Divide the plane into four **quadrants** by the lines XX and YY as before, and let P be any point in the plane (see figure), obtained by laying off a pair of corresponding values of x and y. The position of P is completely determined as soon as x and y are given. Therefore x and y are called the **coördinates** of P, x being called the **abscissa**, and y the **ordinate**.



A point whose coördinates are x and y is referred to as *the point* (x, y).

The four quadrants of the plane are numbered consecutively as in the figure, and are called *the first quadrant, the second quadrant,* and so on.

The line XX is called *the axis of* x, and YY *the axis of* y.

It is evident (definitions of x and y in (**53**)) that the signs of the coördinates in the four quadrants will be as in the following table:

Quadrant	*Abscissa*	*Ordinate*
I	+	+
II	−	+
III	−	−
IV	+	−

57. Linear Equation in Two Variables. — If x and y are unrestricted, the point (x, y) may have any position in the plane. But when a relation between x and y is given, as $y = 2x$ or $y = x + 1$, or $2x - 3y + 4 = 0$, the point (x, y) is thereby restricted to a definite path, which we have already called the graph of the equation.

A relation of the form $\boldsymbol{Ax + By + C = 0}$ is called the *general linear equation in two variables.*

Theorem: The graph of the linear equation $Ax + By + C = 0$ *is a straight line.*

Proof: If $B \neq 0$, we can write $y = -\frac{A}{B}x - \frac{C}{B}$, which has the form $y = ax + b$. Therefore the graph is a straight line when $B \neq 0$.

If $B = 0$, the equation reduces to $Ax + C = 0$, or $x = -\frac{C}{A}$, unless $A = 0$. But if $A = 0$ and $B = 0$, then $C = 0$ and the equation vanishes identically. Excluding this, we may reduce $Ax + C = 0$ to $x = -\frac{C}{A}$, or $x =$ a constant. But this is a straight line parallel to the y-axis. Therefore the given linear equation represents a straight line. (Hence the term "linear" equation.)

Exercises.

1. Show that the equations $Ax + By + C = 0$ and $y = -\frac{A}{B}x - \frac{C}{B}$ have the same graph.

2. Show that the equations $Ax + By + C = 0$ and $kAx + kBy + kC = 0$ have the same graph, k being any constant.

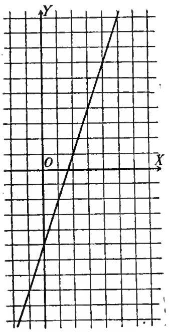
3. How is the graph of $Ax + By + C = 0$ affected by a change in C? In B? in A?

58. Use of the Graph. — When any two variable quantities are connected by a *linear equation*, the relation between them can always be represented graphically by a straight line. It is only necessary to consider the two quantities as the coördinates of a point.

Example 1. A man starts 5 miles south of A and walks due north at the rate of 3 miles an hour. How far is he from A at the end of x hours?

Solution. Let y be the required distance. Also let y be negative to the south of A, positive to the north. Then the relation between y and x is

$$y = 3x - 5.$$



$y = 3x - 5$

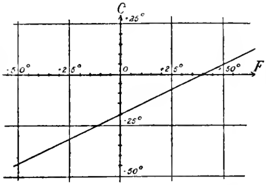
The graph is shown in the figure. Here one square on the horizontal scale represents one hour, and one square on the vertical scale represents one mile.

Exercise. By inspection of the graph, find the distance from A at the end of 0, 2, 3, $4\frac{1}{2}$ hours respectively. Compare with the values obtained from the equation.

In this example negative values of x and the corresponding values of y may be interpreted as follows: Let the time be counted from the moment when the man, supposed to be walking due north continuously at the rate of 3 miles an hour, arrives at the point 5 miles south of A. Let time after this moment be called positive, and before it, negative. Thus, 3 hours before this moment would be represented by $x = -3$. The corresponding value of y is -14, that is, the man was 14 miles south of A.

Example 2. The relation between the readings on the scales of a Centigrade and a Fahrenheit thermometer is given by the equation

$$C = \tfrac{5}{9}(F - 32).$$



Draw the graph.

We shall retain the letters F and C instead of replacing them by x and y. The graph is shown in the adjacent figure. From it the reading of either scale corresponding to a given reading of the other may be at once read off, with an accuracy of about 1°.

Exercise. Read off the values of C corresponding to $F = -40°$, $F = 0°$, $F = 57°$ respectively; also the values of F when $C = -30°$, $0°$, $+21°$.

Example 3. A volume of gas expands when the temperature rises and contracts when the temperature falls according to the law

$$v = v_0 \left(1 + \tfrac{1}{273} t\right),$$

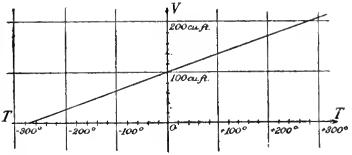
where v_0 = volume at temperature 0°,

and v = volume at temperature t°.

Represent graphically the relation between volume and temperature for a quantity of gas whose volume at 0° is 100 cu. ft.

Replacing $\frac{1}{273}$ by its approximate value .0037, and v_0 by 100, the equation becomes

$$v = .37\,t + 100.$$



The graph is given in the adjacent figure.

59. Exercises.

1. From the figure, read off the volumes corresponding to the temperatures 250°, 75°, 0°, and − 273°; also the temperature corresponding to the volumes 150, 75, and 20 cu. ft. respectively.

2. Construct a graphic conversion table for converting yards to feet.

3. Construct a graph showing the relation between the circumference and the diameter of a circle.

4. A falling body, starting with an initial velocity of v_0 ft. per second, acquires in t seconds a velocity given by $v = gt + v_0$, in which $g = 32.2$. Assume a value of v_0 and draw the graph of the equation.

5. Let A be the lateral area of a right circular cylinder of height h and radius of base r. Draw the graph showing the relation between A and h when r is fixed. Also draw the graph showing the relation between A and r when h is fixed.

6. Same as 5, except that cone is substituted for cylinder, and slant height for height.

Solve for x graphically:

7. $8 + x = 15.$

8. $3\,x = 27.$

9. $2\,(x - 1) = 6.$

10. $\frac{1}{2}x + \frac{1}{3}x = 5.$

11. $\dfrac{x - 2}{3\,x - 5} = \dfrac{6}{19}.$

12. $\dfrac{\frac{3}{4}x - 1}{11 - \frac{2}{3}x} = \dfrac{8}{3}.$

60. Problems.

1. If 12 be added to 7 times a certain number the sum is 54. What is the number?

2. Find a number such that if 16 be subtracted from it and the result multiplied by 5, the product equals the number.

3. Find a number such that if a be subtracted from it and the result multiplied by m, the product equals the number.

4. Find a number such that 3 times the number increased by 10 equals 5 times the number.

5. Find a number such that m times the number increased by a equals n times the number.

6. The age of a boy is three times that of his brother, and their combined ages make 16 years. How old is each?

7. In what proportion must two liquids, of specific gravities 1.20 and 1.40 respectively, be mixed to form a liquid of specific gravity 1.25?

8. Two boys start together and walk around a circular half-mile track at the rates of $3\frac{1}{2}$ and 4 miles an hour respectively. After how many laps will they pass each other?

9. A can do a piece of work in 3 days, B in 5 days. How long will it take them both to do it?

10. A can do a piece of work in a days, B in b days. How long will it take them both to do it?

11. A can do a piece of work in a days, B in b days, and C in c days. In how many days can they together do it?

12. At what time between 4 and 5 are the hands of a clock together?

13. At what time between 10 and 11 are the hands of a clock at right angles? Opposite each other?

14. The sum of the ages of A, B, and C is 60 years. In how many years will the sum be 5 times as great as it was 10 years ago?

15. Water flows into a cistern through two pipes A and B, and out through a third pipe C. The cistern can be filled by A in 1 hour, by B in 45 minutes, and emptied by C in 36 minutes. How long will it take to fill the empty cistern when all three pipes are running?

61. Simultaneous Linear Equations. — Let there be given two linear equations containing two unknown quantities x and y, as

$$ax + by + c = 0,$$
$$a'x + b'y + c' = 0.$$

It is required to obtain all pairs of values of x and y which satisfy both equations simultaneously.

First Method — By Substitution. — Solve one of the equations for either of the unknowns in terms of the other; substitute the value so found in the second equation, thus obtaining a linear equation in one unknown; solve for this unknown and then obtain the other by substitution in either of the given equations.

Check. Substitute the values of x and y in the equation not used in the last step of the solution.

Example. Solve for x and y:

$$\frac{x+y}{3} + x = 15 \quad \text{and} \quad \frac{x-y}{5} + y = 6.$$

Clearing and simplifying:

$$4x + y = 45 \quad \text{and} \quad x + 4y = 30.$$

From the first of these, $y = 45 - 4x$.

Substituting in the second, $x + 4(45 - 4x) = 30$.

Hence $15x = 150$ or $x = 10$.

Then $y = 45 - 4x = 5$.

Check. $\frac{x-y}{5} + y = \frac{10-5}{5} + 5 = 1 + 5 = 6$.

Second Method — By Elimination. — Multiply the first equation by a', the second by $-a$, and add the resulting equations together. This eliminates x, and yields a linear equation in y alone, from which y may be found. Similarly x is found by multiplying the first equation by b', the second by $-b$, and adding. The proper multipliers for the two eliminations are conveniently indicated thus:

$$\begin{array}{r|r|l} b' & a' & ax + by + c = 0, \\ -b & -a & a'x + b'y + c' = 0. \end{array}$$

Check. Substitute the values of x and y in either of the given equations.

Example. Solve for x and y:

$$8x - 15y + 30 = 0 \text{ and } 2x + 3y - 15 = 0.$$

Indicating the multipliers:

$$\begin{array}{r|r|l} 3 & 2 & 8x - 15y + 30 = 0 \\ 15 & -8 & 2x + 3y - 15 = 0 \end{array}$$

$$-54y + 180 = 0, \text{ or } y = \tfrac{10}{3},$$
$$54x - 135 = 0, \text{ or } x = \tfrac{5}{2}.$$

Check. $8 \times \frac{5}{2} - 15 \times \frac{10}{3} + 30 = 20 - 50 + 30 = 0$.

62. Exceptional Cases.

1. *The given equations are not independent.*

In this case one equation is a multiple of the other, so that

$$a = ka', \ b = kb', \text{ and } c = kc',$$

k being a constant. Both equations are then equivalent to a single one, and do not suffice to determine two unknowns.

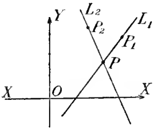
By assuming any value for x, substituting in one of the equations and solving for y, we obtain a pair of values which satisfy *both* equations. (Why?) Hence there exists an infinite number of solutions.

2. *The given equations are inconsistent.*

If $a = ka'$ and $b = kb'$, but $c \neq kc'$, then the given equations are self-contradictory. For if we subtract k times the second equation from the first, we obtain $c = kc'$, which is not true.

In this case there is no finite solution possible. For if we assume $x = x_1$ and $y = y_1$ to be a solution of either equation, the other equation will not be satisfied by these values because $c \neq kc'$.

63. Graphic Solution of Two Simultaneous Linear Equations.



Let the equations be

(1) $$ax + by + c = 0,$$

(2) $$a'x + b'y + c' = 0.$$

The graph of each equation is a straight line. Suppose L_1 and L_2 (figure) to be the graphs of equations (1) and (2) respectively. Then the coördinates of any point on L_1, as P_1, satisfy equation (1), and of any point P_2 on L_2 satisfy (2). Hence *the coördinates of the intersection P of L_1 and L_2 satisfy both equations simultaneously* and give the required solution.

Exceptional Cases.

1. *The given equations are not independent.*

Then, as before, $a = ka'$, $b = kb'$, and $c = kc'$. The lines L_1 and L_2 will coincide and have an infinite number of common points.

2. *The given equations are inconsistent.*

Then $a = ka'$, $b = kb'$, but $c \neq kc'$. The lines L_1 and L_2 are now parallel to each other, but not coincident. Hence they have no common point (except at infinity). Including the infinite solution is equivalent to the statement "parallel lines meet at infinity."

64. Exercises. Solve for x and y, including graphical solutions:

1. $2x + y = 11.$
$3x - y = 4.$

2. $3x + 8y = 19.$
$3x - y = 1.$

3. $2x + y = 47.$
$x + y = 15.$

4. $3x + 4y = 85.$
$5x + 4y = 107.$

5. $5x + 7y = 101.$
$7x - y = 55.$

6. $2x - y - 1 = 0.$
$6x - 3y - 3 = 0.$

7. $15x - 7y = 9.$
$9y - 7x = 13.$

8. $2x - 7y = 8.$
$4y - 9x = 19.$

9. $x - 2y + 2 = 0.$
$3x - 6y + 2 = 0.$

10. $8x + 3y = 3.$
$12x + 9y = 3.$

11. $\frac{2}{3}x + \frac{1}{2}y - 2 = 0.$
$x + \frac{3}{4}y - 3 = 0.$

12. $3y - 4x - 1 = 0.$
$18 - 3x = 4y.$

13. $2x = 11 + 9y.$
$3x - 15 = 12y.$

14. $2x + 7y = 52.$
$3x - 5y = 16.$

15. $3x + 4y - 5 = 0.$
$\frac{1}{2}x + \frac{2}{3}y - \frac{5}{6} = 0.$

16. $5y - 2x = 21.$
$13x - 4y = 120.$

17. $\frac{x}{2} + \frac{y}{3} = 7.$
$2x + 3y = 48.$

18. $\frac{7x}{6} + \frac{5y}{3} = 34.$
$\frac{7x}{8} + \frac{y}{8} = 12.$

19. $4 + y = \frac{3x}{4}.$
$x - 8 = \frac{4y}{5}.$

20. $\frac{2x}{3} = 10 - \frac{1}{2}y.$
$4\frac{3}{4}y = 5x - 7.$

Simultaneous Linear Equations in More than Two Unknowns

65. Three Equations in Three Unknowns.

Let the given equations be,

$$ax + by + cz + d = 0, \tag{1}$$

$$a'x + b'y + c'z + d' = 0, \tag{2}$$

$$a''x + b''y + c''z + d'' = 0. \tag{3}$$

Eliminate one of the variables, say z, from two pairs of the equations, as from (1) and (2) and from (2) and (3). Solve the resulting equations for x and y. Substitute the values of x and y so found in one of the given equations and solve the result for z.

Check. Substitute the values of x, y, and z so found in either of the equations not used in the last step of the solution.

66. Exceptional Cases.

1. *The given equations are not independent.*

(a) In this case one of the equations can be expressed as a linear combination of the other two, with constant coefficients. Hence any solution of these two equations is also a solution of the third. But two equations in three variables admit *an infinity of solutions*. For we can choose any value for z at pleasure, substitute it in the two equations and obtain a pair of values of x and y.

(b) It may happen that two of the equations can be expressed as simple multiples of the third. Then any solution of the third equation is also a solution of the other two. Hence again there exists *an infinity of solutions,* since we may choose for two of the variables any value at pleasure and obtain the corresponding value for the third.

2. *The equations are inconsistent.*

In this case the equations in x and y obtained by eliminating z are also inconsistent. Hence there is *no solution* (except the infinite solution).

67. We shall not discuss here the graphic solution of three linear equations in three variables. Interpreted graphically, each of the equations (1), (2) and (3) represents a plane in space. In general, three planes meet in a single point, giving one and only one solution. The exceptional cases are:

1. (a) The three planes meet in a common line. Hence any point in this line gives a solution.

(b) The three planes coincide. Hence any point in one of the planes is a solution.

2. The three planes are parallel. No solution, except infinity. ("Parallel planes meet at infinity.")

68. Four Equations in Four Unknowns. — *Solution.* Eliminate one of the unknowns from three different pairs of the four given equations. The three resulting equations can be solved for the other three unknowns. The fourth unknown is then found by substituting these three in one of the given equations.

Check. Substitute the values of the four unknowns in one of the equations not used in the last step of the solution.

Exceptional cases arise, quite analogous to the preceding. We shall not discuss them here.

The method of solution outlined above is evidently applicable to any number of linear equations in the same number of variables. A more convenient method involves the use of determinants. (Chapter XVI.)

69. Exercises and Problems.

1. $\frac{x}{4} + \frac{y}{6} = 3\frac{1}{2}.$
$\frac{x}{3} - \frac{y}{8} = \frac{1}{2}.$

2. $\frac{x}{3} + \frac{y}{2} = \frac{4}{3}.$
$\frac{x}{2} + \frac{y}{3} = \frac{7}{6}.$

3. $\frac{x+y}{3}+\frac{y-x}{2}=9.$

$\frac{x}{2}+\frac{x+y}{9}=5.$

4. $\frac{x+y}{8}+\frac{x-y}{6}=5.$

$\frac{x+y}{4}-\frac{x-y}{3}=10.$

5. $\frac{x-1}{8}+\frac{y-2}{5}=2.$

$2x+\frac{2y-5}{3}=21.$

6. $\frac{4x+5y}{40}=x-y.$

$\frac{2x-y}{3}+2y=\frac{1}{2}.$

7. $.25x+3y=10.$
$4.5x-4y=6.$

8. $4.2y+4x=33.$
$0.77y-0.3x=2.95.$

9. $0.2525x+0.33y=280.$
$3.122x+0.055y=3096.$

10. $0.2y+0.25x=2(y-x).$
$0.8x-3.7y=-15.3.$

11. $0.1y+0.3x=0.3.$
$0.05y+0.15x=0.15.$

12. $\frac{3}{4}x-0.6y=0.$

$\frac{5(x-1)}{2(2y+3)}=\frac{5}{6}.$

13. $\frac{1}{x}+\frac{1}{y}=\frac{11}{30}.$

$\frac{1}{x}-\frac{1}{y}=\frac{1}{30}.$

14. $\frac{3}{x}+\frac{8}{y}=3.$

$\frac{15}{x}-\frac{4}{y}=4.$

15. $\frac{1}{x}+\frac{2}{y}=10.$

$\frac{4}{x}+\frac{3}{y}=20.$

16. $\frac{2(5-11x)}{11(x-1)}+\frac{11-7y}{3-y}=5.$

$\frac{7+2x}{3-x}-\frac{125-144y}{36(y+5)}=2.$

17. $\frac{7-6x}{10y-19}=\frac{4-3x}{5y-11}.$

$\frac{6x-10y-17}{3x-5y+2}=\frac{4x-14y-5}{2x-7y+12}.$

18. $\frac{1}{1-x+y}-\frac{1}{x+y-1}=\frac{2}{3}.$

$\frac{1}{\frac{1}{1-x+y}-\frac{1}{1-x-y}}=\frac{3}{4}.$

19. $\frac{1}{x+\frac{1}{y-\frac{5}{x}}}=\frac{1}{x-\frac{1}{y-\frac{7}{x}}}.$

$\frac{1}{y}\left(1-\frac{1}{x}\right)=1.$

20. $ax-by=m.$
$cx+dy=n.$

21. $x+y=3a-2b.$
$x-y=2a-3b.$

22. $\frac{x}{a}+\frac{y}{b}=c.$

$\frac{x}{d}=y.$

23. $\frac{x}{a}-\frac{y}{b}=c.$

$\frac{x}{g}-\frac{y}{f}=0.$

24. $\frac{x}{m}+\frac{y}{n}=p.$

$\frac{x}{r}+\frac{y}{s}=q.$

25. $\frac{mx}{n}+\frac{py}{q}=t.$

$\frac{x}{n}-\frac{ry}{s}=v.$

26. $\frac{x+y-1}{x-y+1}=m.$

$\frac{y-x+1}{x-y+1}=mn.$

27. $\frac{x-m}{y}=\frac{p}{q}.$
$\frac{m}{x}=\frac{n}{y}.$

28. $\frac{a}{b+y}=\frac{b}{a-x}.$
$\frac{c}{d-x}=\frac{d}{c+y}.$

29. $\sqrt{x+1}-\sqrt{y-1}=4.$
$\sqrt{x+1}+\sqrt{y-1}=2.$

30. $2\sqrt{x+y}-y+1=0.$
$\sqrt{x+y}-2y-2=0.$

31. $\frac{1}{\sqrt{x-y}}+\frac{2}{\sqrt{x+y}}+3=0.$
$\frac{3}{\sqrt{x-y}}-\frac{2}{\sqrt{x+y}}+1=0.$

32. $\frac{2}{ax+by}+\frac{x}{y}=5.$
$\frac{3x}{y}-\frac{1}{ax+by}+2=0.$

33. $x+y=37.$
$x+z=25.$
$y+z=22.$

34. $x+y=a.$
$x+z=b.$
$y+z=c.$

35. $2x+3y=12.$
$3x+2z=11.$
$3y+4z=10.$

36. $\frac{1}{5}x-\frac{1}{2}y=0.$
$\frac{1}{3}x-\frac{1}{2}z=1.$
$\frac{1}{2}z-\frac{1}{3}y=2.$

37. $1\frac{1}{3}x+1\frac{1}{2}y=10.$
$2\frac{2}{3}x+2\frac{2}{5}z=20.$
$3\frac{1}{4}y+3\frac{2}{5}z=30.$

38. $x+2y+3z=32.$
$2x+3y+z=42.$
$3x+y+2z=40.$

39. $\frac{1}{y}+\frac{1}{z}=2.$
$\frac{1}{x}+\frac{1}{z}=4.$
$\frac{1}{x}+\frac{1}{y}=6.$

40. $\frac{xy}{x+y}=\frac{1}{5}.$
$\frac{xz}{x+z}=\frac{1}{6}.$
$\frac{yz}{y+z}=\frac{1}{7}.$

41. $x+2y=5.$
$y+2z=8.$
$z+2u=11.$
$u+2x=6.$

42. $y+z+u=2.$
$z+u+x=3.$
$u+x+y=4.$
$x+y+z=5.$

43. $3x+y+z=4.$
$x+4y+3u=6.$
$6x+z+3u=8.$
$8y+3z+5u=10.$

44. Find two numbers whose sum is 1735 and difference 555.

45. If at a given place the longest day exceeds the shortest night by 8 hours 10 minutes, what is the duration of each?

46. The sum of two numbers is 1000. Twice the first plus three times the second equals 2222. Find the numbers.

47. The annual interest on a capital is \$180; at a rate of interest $1\frac{1}{2}\%$ higher, the annual interest would be \$240; find the capital and rate of interest.

48. A farmer sells 200 bushels of wheat and 60 bushels of corn for \$252; 60 bushels of wheat and 200 bushels of corn would bring, at the same price per bushel, \$203; find the price per bushel of each.

49. Two points move on the perimeter of a circle 999 ft. long; the one point, moving four times as fast as the second, overtakes it every 37 seconds; find the speed of each.

50. A vat of capacity 450 cu. ft. can be filled by two pipes. If the first pipe flows 3 minutes and the second 1 minute, 40 cu. ft. are discharged; if the first pipe flows 1 minute and the second 7 minutes, 60 cu. ft. are discharged.

How long will it take both pipes to fill the tank, and what is the discharge per minute of each pipe?

51. How many pounds of copper, and how many of zinc, are contained in 124 pounds of brass (alloy of copper and zinc), if, when placed in water, 89 lbs. of copper lose 10 lbs. in weight, 7 lbs. zinc lose 1 lb., and 124 lbs. brass lose 15 lbs.?

52. An alloy of gold and silver weighing 20 lbs. loses $1\frac{1}{4}$ lbs. when placed in water. How much gold and how much silver does it contain, if gold, when placed in water, loses $\frac{1}{19}$ of its weight, and silver $\frac{1}{10}$ of its weight?

53. Find the lengths of the sides of a triangle if the sum of the first and second is 30, of the first and third 33 and of the second and third 37.

54. Find three numbers which are in the ratio of 2 : 3 : 4 and whose sum is 999.

55. The contents of three measures are as 4 : 7 : 6; 10 measures of the first kind, 4 of the second, and 2 of the third together contain 20 gallons. How much does each measure contain?

56. A vessel may be filled by each of three measures as follows: by 4 of the first and 4 of the third, or by 20 of the first and 20 of the second, or by 28 of the first and 3 of the third. Also, the three measures together contain 29 pints. Find the content of each measure.

57. A vessel can be filled by three pipes: by the first and second in 72 minutes, by the second and third in 2 hrs., and by the first and third in $1\frac{1}{2}$ hrs. How long will it take each pipe alone to fill the vessel?

58. A and B can do a piece of work in 12 days, B and C in 20 days, A and C in 15 days. How long will it take A, B, and C, working together, to do the job?

59. Three principals are placed at interest for a year, A at 4%, B at 5%, C at 6%; the interest on A and B is $796, on B and C $883, and on A and C $819. Find the amount of each principal.

60. Two bodies move on the circumference of a circle; when going in the same direction they meet every 30 seconds, and when going in opposite directions every 10 seconds; in the second case, when they are 30 ft. apart, they will again be 30 feet apart after 3 seconds. Find the speed of each body and the radius of the circle.

CHAPTER V

Quadratic Equations

71. Suppose we wish to find two numbers whose sum is 5 and whose product is 6.

Let $\qquad x =$ one of the numbers;

then $\qquad 5 - x =$ the other number,

and $\qquad x(5 - x) = 6$ or $x^2 - 5x + 6 = 0$.

To determine x we must solve this equation.

Definition. An equation of the form

$$ax^2 + bx + c = 0,$$

where x is a variable and a, b, c are constants, is called the *general equation of the second degree in one variable*, or, *a quadratic equation in x*.

Methods for Solving the Equation $ax^2 + bx + c = 0$.

72. 1. **By Factoring.** When the trinomial $ax^2 + bx + c$ can readily be factored, then each factor, equated to zero, gives a value of x.

Example. $\qquad x^2 - 5x + 6 = 0,$

or $\qquad (x - 2)(x - 3) = 0.$

$\therefore \qquad x - 2 = 0$ or $x - 3 = 0.$

Hence $\qquad x = 2$ or $x = 3.$

73. 2. **By Completing the Square.**

(a) The equation is reduced to the form

$$(x + h)^2 = k,$$

whence $\qquad x + h = \pm\sqrt{k}$, and $x = -h \pm \sqrt{k}$.

This reduction is effected as follows :

Given $\qquad ax^2 + bx + c = 0.$

Transpose c : $\qquad ax^2 + bx = -c.$

Divide by a: $$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Add $\left(\frac{b}{2a}\right)^2$ to both members:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2,$$

or, $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

$$\therefore \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac}.$$

Hence, $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(b) The equation is reduced to the form $(2ax + h)^2 = k$,

whence $$2ax + h = \pm\sqrt{k}, \text{ and } x = \frac{-h \pm \sqrt{k}}{2a}.$$

Given $$ax^2 + bx + c = 0.$$

Transpose c: $$ax^2 + bx = -c.$$

Multiply by $4a$: $$4a^2x^2 + 4abx = -4ac.$$

Add b^2: $$4a^2x^2 + 4abx + b^2 = b^2 - 4ac,$$

or, $$(2ax + b)^2 = b^2 - 4ac.$$

$$\therefore \quad 2ax + b = \pm\sqrt{b^2 - 4ac}.$$

Hence, $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example. $$2x^2 + x - 6 = 0.$$

(a) Transpose -6: $$2x^2 + x = 6.$$

Divide by 2: $$x^2 + \tfrac{1}{2}x = 3.$$

Add $(\frac{1}{4})^2$: $$x^2 + \tfrac{1}{2}x + (\tfrac{1}{4})^2 = 3 + (\tfrac{1}{4})^2,$$

or $$(x + \tfrac{1}{4})^2 = \tfrac{49}{16}.$$

$$\therefore \quad x + \tfrac{1}{4} = \pm\tfrac{7}{4},$$

and $$x = \pm\tfrac{7}{4} - \tfrac{1}{4} = \tfrac{3}{2} \text{ or } -2.$$

(b) Transpose -6: $$2x^2 + x = 6.$$

Multiply by 8: $$16x^2 + 8x = 48.$$

Add 1^2 or 1: $$16x^2 + 8x + 1 = 49,$$

or, $$(4x + 1)^2 = 49.$$

$$\therefore \quad 4x + 1 = \pm 7.$$

Hence $$x = \tfrac{3}{2} \text{ or } -2, \text{ as before.}$$

74. 3. **By Formula.** In (**72**), by completing the square according to either method, we obtained

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Any quadratic equation in x may be solved directly by means of this formula, by merely inserting for a, b, and c their values from the given equation. *The formula should be carefully committed to memory.*

Example. $2x^2 + x - 6 = 0.$

$$x = \frac{-1 \pm \sqrt{1 - 4 \times 2 \times (-6)}}{4} = \frac{-1 \pm 7}{4} = \frac{3}{2} \text{ or } -2.$$

75. Exercises. Solve for x:

1. $x^2 + 4x - 12 = 0.$
2. $x^2 - 8x = -7.$
3. $x^2 + 6x = 16.$
4. $x^2 + 12 = 7x.$
5. $14 = x^2 - 5x.$
6. $5x^2 - 3x - 2 = 0.$
7. $3x^2 + 5x - 42 = 0.$
8. $3x^2 - 50 = 25x.$
9. $2x^2 - 13x = -15.$
10. $3x^2 - 7x - 6 = 0.$
11. $5x(x - 2) + \frac{1}{4} = 1 - 3x.$
12. $(1 - 3x)(x - 6) = 2(x + 2).$
13. $(x + 1)(2x + 3) = 4x^2 - 22.$
14. $7x^2 - 48 = 2x(x + 7).$
15. $13x^2 - 30 = 6(1 - x)^2 + 63.$
16. $ax + b = x^2.$
17. $bx - 2b^2 + x^2 = 2bx.$
18. $x^2 + mn = -(m + n)x.$
19. $cx - 2c - x^2 = -2x.$
20. $x^2 - \frac{ax}{2} = \frac{a^2}{2}.$

76. *Definition.* A *root* of an equation is a value of the variable which satisfies the equation.

By the formula of (**73**), the two roots of any quadratic equation can be obtained.

Nature of the roots of the equation $ax^2 + bx + c = 0$. — The values of x obtained by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

will be

1. **real and unequal** if $b^2 - 4ac > 0$;
2. **real and equal** if $b^2 - 4ac = 0$;
3. **imaginary** if $b^2 - 4ac < 0$.

For, in the first case the radicand in the formula for x is positive, hence the square roots are real; in the second case, the radicand vanishes, and the two values of x reduce to the common value $-b \div 2a$; in the third case the radicand is negative, hence both square roots are imaginary.

The expression $b^2 - 4ac$, on whose value depends the nature of the roots, is called the **discriminant** of the equation $ax^2 + bx + c = 0$.

When the discriminant vanishes, the roots are equal; $ax^2 + bx + c$ *is then a perfect square.*

77. Exercises. — Without solving the equations, determine the nature of the roots of:

1. Exercises 1–10 of (**74**).
2. $4x^2 + 4x + 1 = 0.$
3. $x^2 + x + 1 = 0.$
4. $6x^2 + 2x - 1 = 0.$
5. $9x^2 + 12x + 4 = 0.$
6. $\frac{1}{2}x^2 - \frac{1}{3}x - \frac{1}{4} = 0.$
7. $0.1x^2 + 0.5x + 0.8 = 0.$
8. $1\frac{1}{2}x^2 - 6\frac{1}{3}x + 8\frac{1}{4} = 0.$
9. $\frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{9} = 0.$
10. $0.06x^2 + 0.22x + 0.08 = 0.$

For what values of the literal quantity involved in the following equations will the roots be real and unequal, equal, or imaginary respectively:

11. $x^2 + 2x + c = 0.$
12. $4x^2 + 4x + h = 0.$
13. $3x^2 - 2x - k = 0.$
14. $\frac{1}{2}x^2 - \frac{1}{3}x + 4a = 0.$
15. $x^2 + 2bx + 4 = 0.$
16. $3x^2 - 4kx + 5 = 0.$
17. $6x^2 + \frac{4}{m}x - 3 = 0.$
18. $2x^2 + 4hx - h^2 = 0.$
19. $2x^2 + 4ax - a = 0.$
20. $ax^2 + 2x + 1 = a.$
21. $a^2x^2 + ax + 5 = 0.$
22. $2cx^2 + 3x - c^2 = 0.$
23. $(1 + k)x^2 + x - k = 0.$
24. $\frac{x^2}{n} + \frac{1}{2}x + \frac{1}{n+1} = 0.$

78. Relations between the Coefficients and Roots of a Quadratic Equation. — The roots of the equation

$$ax^2 + bx + c = 0$$

are:
$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Hence
$$x_1 + x_2 = -\frac{b}{a}, \quad \text{and} \quad x_1x_2 = \frac{c}{a}.$$

That is, *if the equation be divided by the coefficient of x^2, the new coefficient of x, with its sign changed, equals the sum of the roots; the new constant term equals the product of the roots.*

79. Factors of the Trinomial $ax^2 + bx + c$. — If x_1 and x_2 be the roots of the equation $ax^2 + bx + c = 0$, the trinomial is divisible by $x - x_1$ and $x - x_2$. But

$$(x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1x_2$$
$$= x^2 + \frac{b}{a}x + \frac{c}{a}.$$

Therefore $a(x - x_1)(x - x_2) = ax^2 + bx + c$.

Hence to factor the trinomial $ax^2 + bx + c$, place it equal to zero and solve for x; subtract each root from x, form the product of these differences and multiply it by a.

80. Exercises.

1. Find the sum and the product of the roots of the equations in exercises 1-10 of (**76**).

Form equations whose roots are:

2. 2, 3; 4, -1; -2, -1.

3. a, $2a$; p, q; $m + n$, $m - n$.

4. $\sqrt{-1}$, $-\sqrt{-1}$; $1 + \sqrt{-1}$, $1 - \sqrt{-1}$; $a + b\sqrt{-1}$, $a - b\sqrt{-1}$.

5-14. Factor the left members of the equations in exercises 1-10 of (**74**).

15-24. Same for exercises 1-10 of (**76**).

25. Show that the equation $y = x^2 + bx + c$ cannot have a fractional root if b and c are integers.

81. Graphic Solution of Quadratic Equations. — In order to solve the equation

$$ax^2 + bx + c = 0,$$

we must find the values of x which reduce the trinomial $ax^2 + bx + c$ to zero. When a, b, c are given numerical values, the required values of x, when real, may be obtained, exactly or approximately, by trial.

Consider, for example, the equation

$$2x^2 + x - 6 = 0.$$

Designate the trinomial on the left by y, so that

$$y = 2x^2 + x - 6.$$

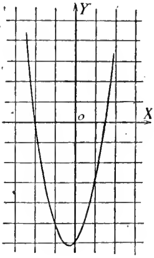
Now form a table showing the values of y corresponding to a series of assumed values of x:

$$x = \ldots -3,\ -2,\ -1,\ \ 0,\ +1,\ +2,\ +3,\ \ldots;$$
$$y = \ldots +9,\ \ 0,\ -5,\ -6,\ -3,\ +4,\ +15,\ \ldots.$$

We see that $y = 0$ when $x = -2$, which gives one root exactly. Also, y must be zero again for a value of x between $+1$ and $+2$, hence the other root lies between 1 and 2.

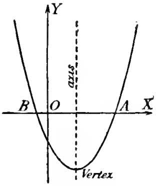
Now consider the pairs of corresponding values of x and y as the coördinates of a series of points and draw a smooth curve through them (figure). Scaling off the values of x for which $y = 0$, we have

$$x = -2 \text{ and } x = 1.5 \text{ approximately.}$$



$y = 2x^2 + x - 6$

82. Parabola. — The curve in the figure is called a **parabola.** It is an example of a class of curves all of which have similar forms. The point where the curve bends most sharply is its **vertex,** and a line through the vertex and dividing the curve into two symmetrical portions is called the **axis** (figure). The segments OA and OB, measured from the origin to the points where the curve cuts the x-axis, are called the **x-intercepts.** The intercepts are positive when extending to the right from O, negative when extending to the left.



Parabola

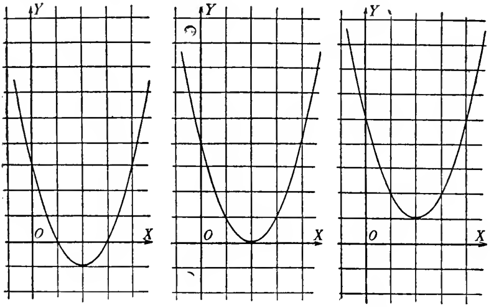
83. It will be found that **the graph of the equation**

$$y = ax^2 + bx + c$$

is always a parabola, with its axis parallel to the y-axis. (We assume $a \neq 0$.)

The parabola will cut the x-axis in two distinct points, or be tangent to the x-axis, or will not cut the x-axis at all according as the equation $ax^2 + bx + c = 0$ has real and unequal, or equal, or imaginary roots. For in the first case y is zero for two distinct values of x, in the second for two equal values of x, and in the third for no real value of x.

These three cases are illustrated in the figures below.



$y = x^2 - 4x + 3$, $b^2 - 4ac > 0$ $y = x^2 - 4x + 4$, $b^2 - 4ac = 0$ $y = x^2 - 4x + 5$, $b^2 - 4ac < 0$

84. Exercises.

1–10. Solve graphically the equations in exercises 1–10 of (**74**).

11–19. Draw the graphs representing the left members of the equations of exercises 2–10 of (**76**).

20. On the same diagram construct the graphs of $y = x^2 + 2x$, $y = x^2 + 2x + 1$, and $y = x^2 + 2x + 2$.

21. Same as in 20 for $y = -x^2 - 2x$, $y = -x^2 - 2x - 1$, and $y = -x^2 - 2x - 2$.

22. What is the effect on the graph of $y = ax^2 + bx + c$ when c is increased or diminished?

23. What is the effect on the graph of changing the signs of all terms of the trinomial?

85. Equations Reducible to Quadratics.

Example 1. $2x^4 - 7x^2 + 6 = 0$.

Solve for x^2 as the unknown quantity.

$$x^2 = \frac{7 \pm \sqrt{49 - 48}}{4} = 2 \text{ or } \frac{3}{2}.$$

$$x = \pm\sqrt{2} \text{ or } \pm\sqrt{\frac{3}{2}}.$$

Example 2. $x^{-\frac{2}{3}} - 8x^{-\frac{1}{3}} = 9$.

Solve for $x^{-\frac{1}{3}}$ as the unknown.

$$x^{-\frac{1}{3}} = \frac{8 \pm 10}{2} = 9 \text{ or } -1.$$

$$x = \frac{1}{729} \text{ or } -1.$$

Example 3. $(2x^2 + 5x)^2 - 6 = 2x^2 + 5x.$

Solve for $2x^2 + 5x$ as the unknown.

$$(2x^2 + 5x)^2 - (2x^2 + 5x) - 6 = 0.$$

$$2x^2 + 5x = \frac{1 \pm 5}{2} = 3 \text{ or } -2.$$

$$2x^2 + 5x = 3 \text{ or } 2x^2 + 5x = -2.$$

$$x = \tfrac{1}{2} \text{ or } -3; \text{ or, } x = -\tfrac{1}{2} \text{ or } -2.$$

Exercise. Verify the answers in the above examples by substitution.

Example 4. $x + \sqrt{2x^2 + 1} = 1.$

Transpose: $\sqrt{2x^2 + 1} = 1 - x.$

Square and collect terms: $x^2 + 2x = 0.$

Therefore $x = 0 \text{ or } -2.$

Example 5. $x - \sqrt{2x^2 + 1} = 1.$

Transpose: $-\sqrt{2x^2 + 1} = 1 - x.$

Square, etc.: $x^2 + 2x = 0.$

Therefore $x = 0$ or -2, as in example 4.

Exercise. Verify the answers in examples 4 and 5.

On substituting the values found in examples 4 and 5 in the given equations, we find that the first equation is satisfied by both values of x, but not the second, provided we assume, as usual, that $\sqrt{2x^2 + 1}$ stands for the positive square root.

The equation of example 5 may be put in the form

$$x - 1 = \sqrt{2x^2 + 1}.$$

Evidently these two expressions are not equal to each other for *any* real value of x. For, if x be less than 1, they are of unlike signs; if x be greater than 1, $\sqrt{2x^2}$ is certainly greater than x, and therefore $\sqrt{2x^2 + 1} > x - 1$. Hence the solution of example 5 as above has led to incorrect results.

The reason for this is that on squaring in the second step of the solution the sign of the radical disappears, and from that point on we are really solving example 4 also.

When an equation is squared to clear of radicals, the answers should be carefully verified and only those retained which satisfy the given condition.

86. Exercises and Problems.

1. $\sqrt{x^2 + 3x - 5} = \sqrt{8x + 1}.$

2. $\sqrt{5x^2 + 1} = \sqrt{3(5x + 7)}.$

3. $\sqrt{2x^2 - 5x + 1} = \sqrt{x + 1}.$

4. $4x - 1 = \sqrt{7x^2 - 2x + 4}.$

5. $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}.$

6. $\sqrt{2x+1}+\sqrt{7x-27}=\sqrt{3x+4}.$

7. $\sqrt{x+3}+\sqrt{3x-3}=10.$

8. $\sqrt{x+17}+\sqrt{x-4}=\frac{7}{4}\sqrt{2x}.$

9. $\sqrt{2x+1}+\sqrt{x-3}=2\sqrt{x}.$

10. $\sqrt{12+x}=\sqrt{7x+8}-2.$

11. $\dfrac{\sqrt{x^2+x+3}}{\sqrt{2x^2+5x-3}}=\dfrac{3}{4}.$

12. $\dfrac{\sqrt{3x^2+x+5}}{\sqrt{4x^2-x+1}}=\dfrac{3}{2}.$

13. $\dfrac{\sqrt{9x^2+6x+1}}{\sqrt{18x^2-3x-2}}=\dfrac{3}{2}.$

14. $\dfrac{\sqrt{9x^2+6x+1}}{\sqrt{18x^2-3x-2}}=-\dfrac{3}{2}.$

15. $\dfrac{4}{y+\sqrt{4-y^2}}+\dfrac{4}{y-\sqrt{4-y^2}}=\dfrac{12}{7}.$

16. $\dfrac{5}{x+\sqrt{x^2+5}}-\dfrac{5}{x-\sqrt{x^2+5}}=6.$

17. $\dfrac{5z-1}{\sqrt{5z}+1}=1+\dfrac{1}{2}(\sqrt{5z}-1).$

18. $\dfrac{4v-1}{\sqrt{2v+1}}+3\sqrt{2v+1}=7\sqrt{v}.$

19. $\dfrac{36}{\sqrt{3s+1}}-\sqrt{5s}=\sqrt{3s+1}.$

20. $\sqrt{7t+4}+\dfrac{11t+15}{\sqrt{4t-3}}=7\sqrt{4t-3}.$

21. $\dfrac{\sqrt{3x^2+1}-\sqrt{2x^2+1}}{\sqrt{3x^2+1}+\sqrt{2x^2+1}}=\dfrac{1}{7}.$

(Or, by composition and division rationalize the denominator.)

22. $\dfrac{\sqrt{27x^2+4}+\sqrt{9x^2+5}}{\sqrt{27x^2+4}-\sqrt{9x^2+5}}=7.$

23. $\dfrac{\sqrt{5x-4}+\sqrt{5-x}}{\sqrt{5x-4}-\sqrt{5-x}}=\dfrac{\sqrt{4x}+1}{\sqrt{4x}-1}.$

24. $\dfrac{\sqrt{2}}{x}=\dfrac{3}{\sqrt{x+x^2}}-\dfrac{\sqrt{2}}{1+x}.$

25. $2\left(1+\dfrac{9}{x}\right)+3\sqrt{\dfrac{x+9}{x}}=14.$

26. $\dfrac{\sqrt{x^2-16}}{\sqrt{x^2-3}}+\sqrt{x+3}=\dfrac{7}{\sqrt{x-3}}.$

27. $\frac{x+m}{x-m}=\frac{p-x}{p+x}.$

28. $\frac{n-x}{n+x}=\frac{x+p}{x-p}.$

29. $\frac{a^2}{x}-\frac{x}{b^2}=\frac{x}{a^2-b^2}.$

30. $\frac{ab-x}{b-ax}=\frac{b-cx}{bc-x}.$

31. $\frac{x^5+a^5}{x+a}+\frac{x^5-a^5}{x-a}=2x^4.$

32. $\frac{\sqrt{a^2-x^2}}{x}=\frac{x}{b}-\frac{a}{x}.$

33. $\frac{y^4-m^4}{y-m}-\frac{y^4-m^4}{y+m}=10m^3.$

34. $\sqrt{a+x}-\sqrt{a-x}=\sqrt{a}.$

35. $\frac{m-\sqrt{2my-y^2}}{m+\sqrt{2my-y^2}}=a.$

36. $\sqrt{x}+\sqrt{2a-x}=\frac{a}{\sqrt{x}}.$

37. $\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}=\frac{\sqrt{a-x}+\sqrt{b-x}}{\sqrt{a-x}-\sqrt{b-x}}.$

38. $\sqrt{x}+\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}.$

39. $\sqrt{a-x}+\sqrt{-(a^2+ax)}=\frac{a}{\sqrt{a-x}}.$

40. $\frac{\sqrt{a-x}+\sqrt{x-b}}{\sqrt{a-x}-\sqrt{x-b}}=\sqrt{\frac{a-x}{x-b}}.$

41. $2x\sqrt[3]{x}-3x\sqrt[3]{\frac{1}{x}}=20.$

42. $\frac{a-x}{\sqrt{a-x}}+\frac{x-b}{\sqrt{x-b}}=\sqrt{a-b}.$

43. $\sqrt{\frac{a-x}{b+x}}+\sqrt{\frac{b+x}{a-x}}=c.$

44. $\sqrt{\frac{a-x}{b-x}}-\sqrt{\frac{b-x}{a-x}}=c.$

45. $x^{\frac{10}{3}}-16x^{\frac{5}{3}}=512.$

46. $x^{2m}+2ax^m=8a^2.$

47. $x^2+\sqrt{5x+x^2}=42-5x.$

48. $\sqrt[3]{x}-5\sqrt[3]{x^2}=-18.$

49. $7\sqrt[3]{-x}+\sqrt[3]{x^2}=-12.$

50. $x^2+24=7x-\sqrt{x^2-7x+18}.$

51. $\sqrt[m]{(1+x)^2}-\sqrt[m]{(1-x)^2}=\sqrt[m]{1-x^2}.$

52. Find three consecutive integers, the sum of whose squares is 1202.

53. Find three consecutive even integers, the sum of whose squares is 776.

54. The sum of the squares of three consecutive integral multiples of 4 is 3104. Find the numbers.

55. A rectangle, twice as long as it is wide, has an area of 1800 square feet. Find its dimensions.

56. How large a square must be cut from each corner of a rectangular card 6 × 12 inches so that the remaining piece shall contain 27 square inches ?

57. As in 56, except that the original dimensions are $a \times b$ inches and the remaining area A square inches.

58. What changes must be made in the dimensions of a rectangle 2 × 12 inches to double the area without changing the perimeter ?

59. As in 58, when the original dimensions are $a \times b$ inches.

60. State some values of a and b for which exercise 59 is impossible.

61. Find the radius of a cylinder whose height is 10 feet, if the total surface in square feet must equal the volume in cubic feet.

62. As in 61, except that total surface equals twice the volume.

63. As in 61, except that total surface equals n times the volume. For what values of n is the problem impossible?

64. What number exceeds twice its square root by 3?

65. The sum of the ages of a father and his son is 80 years and the product of their ages is 15 times the sum; find the age of each.

66. A number consisting of two equal digits is 3 less than 4 times the square of one of its digits; find the number.

67. For what real values of x is $x^2 + 10x + 9$ positive? zero? negative? (Graph.)

68. Show that $6 + 2a + a^2$ cannot be negative if a is real. (Graph.)

69. Show that $3a - a^2 - 5$ cannot be positive if a is real. (Graph.)

70. The difference of the cubes of two consecutive integers is 127. What are the integers?

71. Two trains start from a station, one going due north 5 miles an hour faster than the other, which goes west; at the end of four hours they are 60 miles apart. Find the speed of each.

87. Simultaneous Quadratics.

Definition. The *degree of a monomial* involving one or more literal quantities is the sum of the exponents of such literal quantities as may be specified.

For example $a^p x^m y^n$ is of degree m in x, n in y, $m + n$ in x and y, $m + n + p$ in a, x and y.

The *degree of a polynomial* is that of its term of highest degree.

A *quadratic equation in several variables* is one in which all the variable terms are of the first or second degree, at least one term of the second degree being actually present.

88. Solution of Two Simultaneous Equations in Two Variables, one being Linear, the other Quadratic. — The most general forms of such equations are:

(1) $px + qy + r = 0,$

(2) $ax^2 + by^2 + cxy + dx + ey + f = 0.$

Solution.

1. Solve (1) for one of the variables in terms of the other. Thus:

$$y = -\frac{px + r}{q}.$$

2. Substitute this value in (2), obtaining a quadratic equation in x.

3. Solve this quadratic for x, and let its roots be x_1 and x_2.

4. The corresponding values of y are now found by substituting these values for x in the first step. Thus:

$$y_1 = -\frac{px_1 + r}{q}, \quad \text{and} \quad y_2 = -\frac{px_2 + r}{q}.$$

Example.
$$\text{(a)} \quad x + y = 1,$$
$$\text{(b)} \quad x^2 + y^2 = 4.$$

From (a), $y = 1 - x.$

Substituting in (b): $x^2 + (1 - x)^2 = 4$ or $2x^2 - 2x - 3 = 0.$

Hence $x_1 = \frac{1}{2} + \frac{1}{2}\sqrt{7};\ x_2 = \frac{1}{2} - \frac{1}{2}\sqrt{7}.$

Then $y_1 = \frac{1}{2} - \frac{1}{2}\sqrt{7};\ y_2 = \frac{1}{2} + \frac{1}{2}\sqrt{7}.$

Reducing to decimals, we have approximately

$$(x_1, y_1) = (+1.8, -0.8) \quad \text{and} \quad (x_2, y_2) = (-0.8, +1.8).$$

In this case there are two distinct real solutions.

89. Nature of the Solutions of Equations (1) and (2) of (88). — The values x_1 and x_2 obtained in the third step of the solution in (**88**) are either real and unequal, real and equal, or both imaginary. Then the values of y obtained in the fourth step will be of the same nature as the values of x.

Hence there are always two solutions, which may be real and unequal, real and equal, or imaginary.

90. These three cases may be illustrated by means of the equations,

(1) $$x + y = k,$$

(2) $$x^2 + y^2 = 4.$$

Then $x^2 + (k - x)^2 = 4$, or $2x^2 - 2kx + (k^2 - 4) = 0.$

Hence $x_1 = \frac{1}{2}(k + \sqrt{8 - k^2})$ and $x_2 = \frac{1}{2}(k - \sqrt{8 - k^2}).$

$y_1 = \frac{1}{2}(k - \sqrt{8 - k^2})$ and $y_2 = \frac{1}{2}(k + \sqrt{8 - k^2}).$

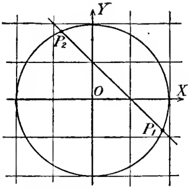
These solutions will be

real and unequal if $k^2 < 8$;
real and equal if $k^2 = 8$;
imaginary if $k^2 > 8$.

91. Graphic Solution of the equations

(1) $$x + y = 1,$$

(2) $$x^2 + y^2 = 4.$$



Straight line $x + y = 1$
Circle $x^2 + y^2 = 4$

Considering x and y as the coördinates of a variable point, all values of x and y which satisfy the equation (1) give rise to a series of points lying on a straight line (figure).

Let us now mark some points whose coördinates satisfy equation (2), which we put into the form

$$y = \pm \sqrt{4 - x^2}.$$

Assuming a set of values for x, and calculating the corresponding values of y, we have

$$x = 0, \tfrac{1}{2}, 1, 1\tfrac{1}{2}, 2, 2\tfrac{1}{2}, \ldots ;$$

$$y = \pm 2, \pm \tfrac{1}{2}\sqrt{15}, \pm \sqrt{3}, \pm \tfrac{1}{2}\sqrt{7}, 0, \text{imaginary}.$$

For negative values of x we obtain the same values of y over again.

On plotting these values we obtain a series of points all of which lie on a *circle* of radius 2, center at the origin.

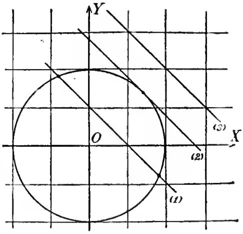
The points of intersection of the line and the circle have coördinates which satisfy both equations at once, and are therefore the required solutions. Scaling them off from the figure we have

$(x_1, y_1) = (1.8, -0.8)$ and $(x_2, y_2) = (-1.8, +0.8)$, as in **(88)**.

92. Graphic Illustration of the Three Cases of (90). — In (1) of **(90)**, let us put successively $k = 1, 2\sqrt{2}$, and 4, so that $k^2 < 8$, $= 8$, and > 8 respectively. We have then the equations,

(1) $x + y = 1;\ x + y = 2\sqrt{2};\ x + y = 4,$
(2) $x^2 + y^2 = 4;\ x^2 + y^2 = 4;\ x^2 + y^2 = 4.$

The three straight lines and the circle are shown in the adjacent figure. When $k = 1$, the line cuts the circle in two distinct points; when $k = 2\sqrt{2}$, the line is tangent to the circle; when $k = 4$, the line fails to meet the circle. We may consider these three cases as arising from special positions of a variable line which moves parallel to itself and occupies in turn the positions of the three lines in the figure.

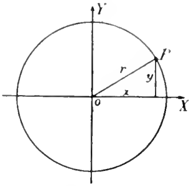


Circle $x^2 + y^2 = 4$

93. Standard Equation of the Circle. — The equation

$$x^2 + y^2 = r^2$$

is satisfied by the coördinates of every point on a circle of radius r, center at the origin, and by no other point. It is called *the standard equation of the circle.*



$x^2 + y^2 = r^2$
Circle, radius r, center at origin

Exercises. Solve for x and y, and check carefully by graphs.

1. $\begin{cases} x^2 + y^2 = 1, \\ x - y = 0. \end{cases}$
2. $\begin{cases} x^2 + y^2 = 1, \\ x - y = 2. \end{cases}$
3. $\begin{cases} x^2 + y^2 = 1, \\ x - y = \sqrt{2}. \end{cases}$
4. $\begin{cases} x^2 + y^2 = 4, \\ 2x + y = 2. \end{cases}$
5. $\begin{cases} x^2 + y^2 = 4, \\ x - 2y = 3. \end{cases}$
6. $\begin{cases} x^2 + y^2 = 16, \\ 2x - 3y = 4. \end{cases}$
7. $\begin{cases} x^2 + y^2 = 9, \\ 3x + 4y = 12. \end{cases}$
8. $\begin{cases} x^2 + y^2 = 9, \\ 4x - 5y = 20. \end{cases}$
9. $\begin{cases} 4x^2 + 4y^2 = 1, \\ 3x - y = 1. \end{cases}$

10. Determine k so that the line $x + y = k$ shall be tangent to the circle $x^2 + y^2 = 4$.

11. Determine m so that the line $y = mx + 5$ shall touch the circle $x^2 + y^2 = 5$.

12. As in 11, for the line $y = mx + 2$ and the circle $x^2 + y^2 = \frac{2}{5}$.

94. Consider the equations

$$x - y = 1,$$
$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

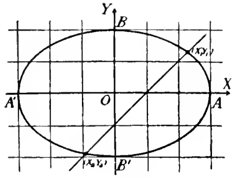
Proceeding as in (**88**), we obtain

$$x_1 = \frac{9 + 12\sqrt{3}}{13} = 2.3 - ; \quad x_2 = \frac{9 - 12\sqrt{3}}{13} = -0.9 - ;$$
$$y_1 = \frac{-4 + 12\sqrt{3}}{13} = 1.3 - ; \quad x_2 = \frac{-4 - 12\sqrt{3}}{13} = -1.9 - .$$

Graphic Solution. — All values of x and y which satisfy the first equation are the coördinates of points on a straight line. We now plot a series of points whose coördinates satisfy the second equation, which we solve for y in terms of x and write in the form

$$y = \pm \tfrac{1}{3}\sqrt{36 - 4x^2}.$$

When $x = -3, \quad -2, \quad -1, \quad 0, \quad +1, \quad +2, \; +3,$
$y = \quad 0, \pm \frac{2}{3}\sqrt{5}, \pm \frac{4}{3}\sqrt{2}, \pm 2, \pm \frac{4}{3}\sqrt{2}, \pm \frac{2}{3}\sqrt{5}, \quad 0.$



Ellipse $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$

Straight line $x - y = 1$

On plotting these points and drawing a smooth curve through them we obtain the curve in the adjacent figure, called an **ellipse.** The line $A'A$ is called the **major axis** of the ellipse, $B'B$ the **minor axis**, and O is the **center.** In this case, $A'A = 6$ and $B'B = 4$; $OA = 3$ and $OB = 2$.

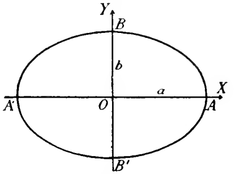
Scaling off the coördinates of the points of intersection of the two graphs, we have as our graphic solution

$$(x_1, y_1) = (2.3, 1.3);\ (x_2, y_2) = (-0.9, -1.9).$$

95. Standard Equation of the Ellipse. — Every equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

represents an **ellipse,** whose major axis is $2a$, minor axis $2b$, center at the origin. It is called the *standard equation of the ellipse.*



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Ellipse, semi-axes a and b respectively

Exercises. Solve and check by graphs:

1. $\begin{cases} \frac{x^2}{9} + \frac{y^2}{4} = 1, \\ x + y = 0. \end{cases}$

2. $\begin{cases} \frac{x^2}{9} + \frac{y^2}{4} = 1, \\ x + y = 5. \end{cases}$

3. $\begin{cases} \frac{x^2}{9} + \frac{y^2}{4} = 1, \\ x + y = \sqrt{13}. \end{cases}$

4. $\begin{cases} 4x^2 + 9y^2 = 36, \\ x + y = -\sqrt{13}. \end{cases}$

5. $\begin{cases} \frac{x^2}{4} + y^2 = 1, \\ x + 2y = 2. \end{cases}$

6. $\begin{cases} x^2 + 4y^2 = 4, \\ x - y = 3. \end{cases}$

7. $\begin{cases} 9x^2 + 16y^2 = 25, \\ 2x - 3y = 6. \end{cases}$

8. $\begin{cases} 2x^2 + 3y^2 = 12, \\ 3x + y = 2. \end{cases}$

9. $\begin{cases} 9x^2 + 4y^2 = 1, \\ x - y = 1. \end{cases}$

10. Determine k so that the line $x - y = k$ shall be tangent to the ellipse $x^2 + 4y^2 = 4$.

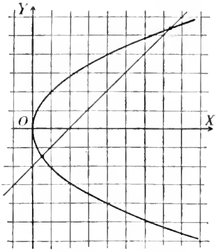
11. Determine m so that the line $y = mx + 3$ shall touch the ellipse $4x^2 + 9y^2 = 36$.

96. Consider the equations

$$x - y = 2,$$
$$y^2 = 4x.$$

Solving as in (88), we find

$$x_1 = 4 + 2\sqrt{3}, \qquad x_2 = 4 - 2\sqrt{3},$$
$$y_1 = 2 + 2\sqrt{3}, \qquad y_2 = 2 - 2\sqrt{3}.$$



Parabola, $y^2 = 4x$

The graphs are shown in the figure, that of the equation $y^2 = 4x$ being a **parabola**, whose vertex is at the origin and whose axis is the x-axis.

Exercise 1. Compare the graphic solution with that obtained by formula.

Exercise 2. For what value of k will the line $x - y = k$ be tangent to the parabola $y^2 = 4x$? Why are there not two values of k as in the exercise of (95)?

97. Standard Equations of the Parabola. — The equation

$$y^2 = 4ax$$

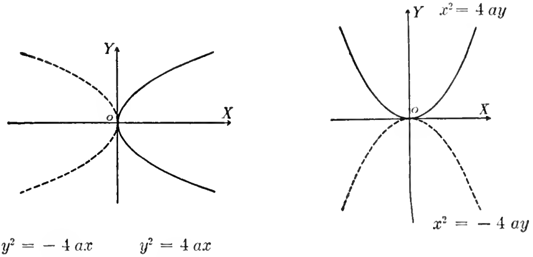
always represents a **parabola**, whose vertex is at the origin and whose axis is the x-axis. The curve extends to the right from O when a is positive, to the left when a is negative.

The equation

$$x^2 = 4ay$$

always represents a **parabola**, whose vertex is at the origin and whose axis is the y-axis. The curve extends upward when a is positive, downward when a is negative.

Parabolas



Exercises. Solve and check by graphs:

1. $\begin{cases} y^2 = x, \\ y = x. \end{cases}$

2. $\begin{cases} y^2 = 4x, \\ x + y = 1. \end{cases}$

3. $\begin{cases} 4y^2 = x, \\ 2x - y = 4. \end{cases}$

4. $\begin{cases} y^2 = -x, \\ y - x = 2. \end{cases}$

5. $\begin{cases} y^2 = -4x, \\ 3x + y = 3. \end{cases}$

6. $\begin{cases} x^2 = y, \\ y = 2x. \end{cases}$

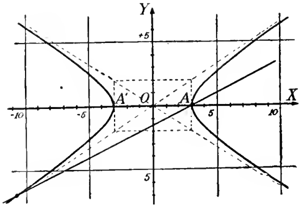
7. $\begin{cases} x^2 = 4y, \\ x + 2y = 2. \end{cases}$

8. $\begin{cases} x^2 = -y, \\ 2x + 5y = 10. \end{cases}$

9. $\begin{cases} x^2 = -4y, \\ y - 2x = 1. \end{cases}$

10. Determine k so that the line $3x + y = k$ shall touch the parabola $y^2 + 4x = 0$.

11. Determine m so that the line $y = mx + 2$ shall touch the parabola $y^2 = 8x$.



Hyperbola, $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$

Straight Line, $x - 2y = 3$

98. Consider the equations

$$x - 2y = 3,$$

$$\frac{x^2}{9} - \frac{y^2}{4} = 1.$$

The graphs are shown in the figure.

The graph of the second equation is an **hyperbola**, a curve consisting of two open branches which continually approach the diagonals, produced, of the dotted rectangle, but never cross them. These lines are called the **asymptotes** of the hyperbola. O is the center and $A'A$ the axis of the curve.

Exercise. Compare the solution of given equation as obtained by formula with that from the graph.

99. Standard Equation of the Hyperbola. — The equation

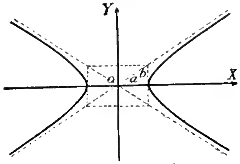
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

always represents an hyperbola whose axis coincides with the x-axis, and whose center is at the origin. The curve lies between its asymptotes, which are the diagonals, produced, of a rectangle whose sides are $2a$ and $2b$, parallel to the coördinate axes, with its center at the origin.

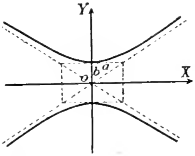
The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

represents an hyperbola whose axis coincides with the y-axis.



Hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

Exercises. Solve and check by graphs:

1. $\begin{cases} x^2 - y^2 = 1, \\ x - 3y = 1. \end{cases}$

2. $\begin{cases} x^2 - y^2 = 1, \\ 2x - y = 1. \end{cases}$

3. $\begin{cases} x^2 - y^2 = 1, \\ \sqrt{5}x - y = 2. \end{cases}$

4. $\begin{cases} \frac{x^2}{9} - \frac{y^2}{4} = 36, \\ 5x + y = 5. \end{cases}$

5. $\begin{cases} 4x^2 - 9y^2 = 36, \\ 4x + y = 2. \end{cases}$

6. $\begin{cases} x^2 - 4y^2 = 4, \\ y = 2x - 6. \end{cases}$

7. $\begin{cases} 2x^2 - 3y^2 = 6, \\ 3x + y = 6. \end{cases}$

8. $\begin{cases} x^2 - y^2 = -1, \\ y - 3x = 1. \end{cases}$

9. $\begin{cases} 4x^2 - 9y^2 = -36, \\ 2y - x = 0. \end{cases}$

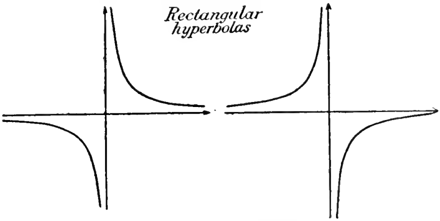
10. Determine k so that the line $x - 2y = k$ shall be tangent to the hyperbola $4x^2 - 9y^2 = 36$.

11. Determine m so that the line $y = mx - 2$ shall touch the hyperbola $x^2 - y^2 = 1$.

100. Rectangular Hyperbola.— The equation

$$xy = \pm k^2$$

always represents an hyperbola whose asymptotes are the coördinate axes; for the upper sign, its branches lie in the first and third quadrants, and for the lower sign in the second and fourth quadrants.



Rect. Hyp. $xy = k^2$ *Rect. Hyp.* $xy = -k^2$

101. The general equation of the second degree,

$$ax^2 + by^2 + cxy + dx + ey + f = 0,$$

includes all the types of equation considered in the preceding sections and always represents one of the curves there shown, except in isolated cases when it can be factored into linear factors, in which case it represents a pair of straight lines, or when it is satisfied by the coördinates of a single point only, as $x^2 + y^2 = 0$. The graph may also be imaginary, that is, the equation cannot be satisfied by any real values of x and y, as $x^2 + y^2 = -1$.

The curves represented by the general equation of the second degree are not restricted in position with respect to the coördinate axes as are those shown in the preceding figures. The center, vertices, axes and asymptotes may have any position whatever, depending on the numerical values of the coefficients a, b, c, d, e.

All curves represented by equations of the second degree in x and y may be obtained as plane sections of a circular cone. They are therefore called **conic sections**.

102. Exercises. Give what facts you can about the curves represented by the following equations, without drawing the graphs:

1. $x^2 + y^2 = 9.$

2. $4x^2 + 4y^2 = 16.$

3. $3x^2 + 3y^2 = 15.$

4. $4x^2 + y^2 = 4.$

5. $x^2 + 4y^2 = 4.$

6. $16x^2 + 25y^2 = 400.$

7. $25x^2 + 16y^2 = 400.$

8. $2x^2 + 4y^2 = 9.$

9. $y^2 = 4x.$

10. $4y^2 = x.$

11. $x^2 = 4y.$

12. $4x^2 = y.$

13. $y^2 = -4x.$

14. $-4y^2 = x.$

15. $x^2 = -4y.$

16. $4x^2 = -y.$

17. $16x^2 - 25y^2 = 400.$

18. $16x^2 - 25y^2 = -400.$

19. $25x^2 - 16y^2 = 400.$

20. $25x^2 - 16y^2 = -400.$

Construct the graphs of the preceding equations on cross-section paper.

Construct the graphs of the equations:

21. $x^2 + y^2 - 6x - 8y = 0.$

22. $(x - y)^2 = 1.$

23. $3x^2 + 2xy + 3y^2 - 16y + 23 = 0.$

24. $x^2 - 5xy + 6y^2 = 0.$

25. $3x^2 + 2y^2 - 2x + y - 1 = 0.$

26. $x^2 + 2xy + y^2 = 0.$

27. $5x^2 + 2xy + 5y^2 = 0.$

28. $4xy + 6x - 8y + 1 = 0.$

29. $y^2 - xy - 5x + 5y = 0.$

30. $xy - y^2 = 1.$

Solve graphically and by formula several of the preceding equations with the equation

(a) $x - y = 1.$ (b) $2x + 3y = 6.$

(c) $x + y = 0.$ (d) $2x - y = 2.$

103. Solution of Two Simultaneous Quadratics. — When both quadratics are of the general form, as

$$ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$a'x^2 + b'y^2 + c'xy + d'x + e'y + f' = 0,$$

they cannot usually be solved by elementary methods. For, if we solve one equation for y in terms of x say, and substitute in the other, we obtain, after rationalizing, an equation of the fourth degree in x. Such an equation requires rather complicated processes for its solution. We shall therefore leave aside the general case and discuss some special cases, such as usually arise in the practical application of algebra. We begin with some graphic illustrations.

104. Graphic Solution. — Since each of the above equations represents graphically a conic section, two such curves intersect in general in four points. All **real** solutions are shown by the intersections of the graphs, and may be read off, approximately at least, from the diagram.

When the graphs intersect in less than four points (tangency is counted as two coincident points of intersection), some solutions are **imaginary** *or* **infinite.**

The various cases which may arise are illustrated in the figures on page 74.

We proceed to consider some special cases of simultaneous quadratic equations.

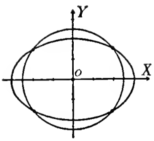
105. Case 1. Two quadratics, one of which is factorable.

Rule: Factor the equation, put each factor equal to zero, and solve each of the resulting linear equations with the other quadratic.

Rule for factoring a quadratic. Solve for y in terms of x (or x in terms of y); if the quantity under the radical is a perfect square the two values of y are of the form $y = ax + b$ and $y = a'x + b$. The required factors are then

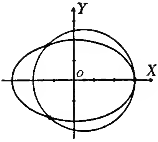
$$(y - ax - b)\,(y - a'x - b').$$

Graphically, the factorable quadratic represents a pair of straight lines, the other quadratic some conic. Each straight line may cut this conic in two real distinct points, in two real coincident points, or in two imaginary points (i.e. does not cut at



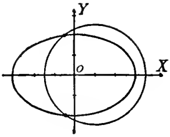
$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$
$$x^2 + y^2 = \tfrac{25}{4}$$

Four real solutions, all distinct.



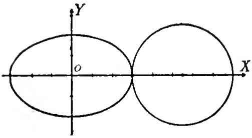
$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$
$$(x - \tfrac{1}{2})^2 + y^2 = \tfrac{25}{4}$$

Four real solutions, two being equal.



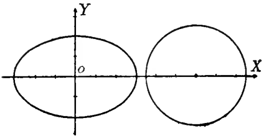
$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$
$$(x - 1)^2 + y^2 = \tfrac{25}{4}$$

Two real distinct solutions, two imaginary.



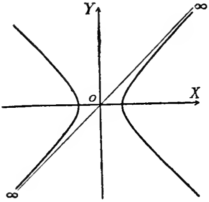
$$\frac{x^2}{9} + \frac{y^2}{4} = 1; \quad \left(x - \frac{11}{2}\right)^2 + y^2 = \frac{25}{4}$$

Two real and equal solutions, two imaginary.



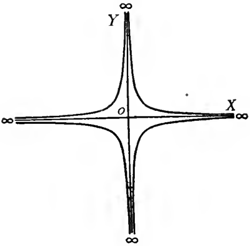
$$\frac{x^2}{9} + \frac{y^2}{4} = 1; \quad (x - 6)^2 + y^2 = \frac{25}{4}$$

All four solutions imaginary.



$$x^2 - y^2 = 1$$
$$x - y = 0$$

Two solutions, both infinite.



$$xy = 1$$
$$xy = -1$$

Four solutions, all infinite.

The student is urged to draw, or to picture to himself mentally as far as possible, graphs corresponding to all equations considered. He should be able to recognize at a glance the standard forms of equation of the conic sections.

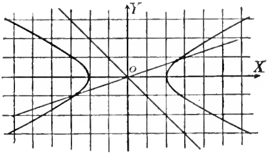
all). Hence the four solutions may be all real and distinct, or equal in pairs, or imaginary in pairs.

Example 1. $\begin{cases} x^2 - 2xy - 3y^2 = 0, \\ x^2 - 4y^2 - 4 = 0. \end{cases}$

The factors of the first equation are, by inspection,

$$(x + y)(x - 3y) = 0.$$

$$\therefore \quad x + y = 0 \quad \text{or} \quad x - 3y = 0.$$



Hyperbola, $x^2 - 4y^2 - 4 = 0$
Straight lines, $x^2 - 2xy - 3y^2 = 0$
or $x + y = 0$ *and* $x - 3y = 0$

Hence we have to solve

$$\begin{cases} x + y = 0, \\ x^2 - 4y^2 - 4 = 0, \end{cases} \quad \text{and} \quad \begin{cases} x - 3y = 0, \\ x^2 - 4y^2 - 4 = 0. \end{cases}$$

Solving the first pair, we have

$$(x_1, y_1) = \left(\frac{4}{\sqrt{-3}}, -\frac{4}{\sqrt{-3}}\right); \quad (x_2, y_2) = \left(-\frac{4}{\sqrt{-3}}, \frac{4}{\sqrt{-3}}\right).$$

These are imaginary. The line $x + y = 0$ does not cut the hyperbola (figure).

Solving the second pair,

$$(x_3, y_3) = \left(\frac{6}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right); \quad (x_4, y_4) = \left(-\frac{6}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right).$$

These solutions are real, and the approximate values may be scaled off from the figure.

Note. An equation of the form $Ax^2 + Bxy + Cy^2 = 0$ *can always be factored.* Divide by the square of one of the variables, and solve for the ratio $\frac{y}{x}$ or $\frac{x}{y}$.

The factors will be imaginary if $B^2 - 4AC < 0$, and in this case the graph of the equation is imaginary. In all other cases the graph is a pair of real straight lines, distinct if $B^2 - 4AC > 0$, and coincident if $B^2 - 4AC = 0$.

Example 2. Factor $2x^2 - 2xy + y^2 = 0$.

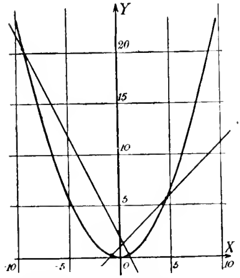
Divide by x^2: $\left(\frac{y}{x}\right)^2 - 2\left(\frac{y}{x}\right) + 2 = 0.$

$$\therefore \quad \frac{y}{x} = 1 + \sqrt{-1} \text{ or } 1 - \sqrt{-1}.$$

Hence the factors are

$$[y - (1 + \sqrt{-1})x][y - (1 - \sqrt{-1})x] = 0.$$

Example 3. $\begin{cases} 2x^2 - y^2 - xy + 3y - 2 = 0. \\ x^2 - 4y = 0. \end{cases}$



Parabola, $x^2 - 4y = 0$
Straight lines,
$2x^2 - y^2 - xy + 3y - 2 = 0$
or $x - y + 1 = 0$
and $2x + y - 2 = 0$

Solving the first equation for x in terms of y, we have

$$x = \frac{y \pm \sqrt{9y^2 - 24y + 16}}{4} = \frac{y \pm (3y - 4)}{4}.$$

Hence,

$$x - y + 1 = 0 \quad \text{or} \quad 2x + y - 2 = 0.$$

Solving the first of these with the second equation above, we have

$$(x_1, y_1) = (2 + 2\sqrt{2}, 3 + 2\sqrt{2});$$
$$(x_2, y_2) = (2 - 2\sqrt{2}, 3 - 2\sqrt{2}).$$

From the second equation we obtain

$$(x_3, y_3) = (-4 + 2\sqrt{6}, 10 - 4\sqrt{6});$$
$$(x_2, y_2) = (-4 - 2\sqrt{6}, 10 + 4\sqrt{6}).$$

Exercises. Solve for x and y, and check graphically:

1. $\begin{cases} x^2 + y^2 = 1, \\ x^2 + yx - 2y^2 = 0. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 4, \\ x^2 - y^2 = 0. \end{cases}$

3. $\begin{cases} 4x^2 + 9y^2 = 36, \\ 2x^2 + 5xy + 3y^2 = 6x + 6y. \end{cases}$

4. $\begin{cases} x^2 - y^2 = 1, \\ xy - 2y + x = 2. \end{cases}$

5. $\begin{cases} y^2 - 4x = 0, \\ 6x^2 + xy - 12y^2 = 0. \end{cases}$

6. $\begin{cases} x^2 - 4y^2 = 4, \\ xy - 2y = 0. \end{cases}$

106. Case 2. Homogeneous equations.

Definition. An equation is called *homogeneous* when all of its variable terms are of the same degree. A constant term may be present. (In the further developments of mathematics, the last sentence is omitted from the definition.)

Two homogeneous quadratics have the forms

$$(1) \qquad Ax^2 + Bxy + Cy^2 = D,$$
$$(2) \qquad A'x^2 + B'xy + C'y^2 = D'.$$

Solution. Multiply the first equation by D', the second by D and subtract. The result is a new equation of the form

$$(3) \qquad A''x^2 + B''xy + C''y^2 = 0,$$

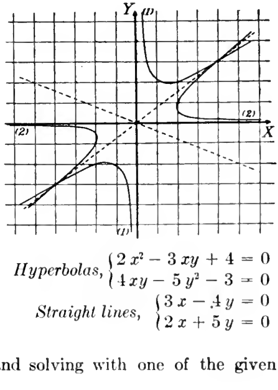
which may be solved with either of the given equations by factoring, as in *Case* 1.

Graphically, equations (1) and (2) represent two conics, and equation (3) a third conic which consists of a pair of straight lines in case the factors are real. Conic (3) goes through the intersections of (1) and (2), since the coördinates of any point which satisfy (1) and (2) will also satisfy (3). Hence, when the factors of (3) are real, we obtain the intersections of (1) and (2) by finding the intersections of either of them with a pair of real straight lines. When these factors are distinct, there are two distinct lines, either of which may cut the conic in two real and distinct points, two coincident points, or two imaginary points. When the factors are imaginary the lines are imaginary, and all four solutions are imaginary.

Another method of solving two homogeneous equations in the forms (1) and (2) is to put in both of them $y = vx$. Then divide one equation by the other, and clear of fractions, after removing the common factor x^2. The result is a quadratic in v, whose roots we may represent by v_1 and v_2. Then

$$y = v_1 x \quad \text{and} \quad y = v_2 x.$$

Substituting these values in turn in either of the given equations, we have two quadratic equations in x alone.



Hyperbolas, $\begin{cases} 2x^2 - 3xy + 4 = 0 \\ 4xy - 5y^2 - 3 = 0 \end{cases}$

Straight lines, $\begin{cases} 3x - 4y = 0 \\ 2x + 5y = 0 \end{cases}$

Example 1. $\begin{cases} 2x^2 - 3xy + 4 = 0, \\ 4xy - 5y^2 - 3 = 0. \end{cases}$

Transposing the constant terms we have

$$2x^2 - 3xy = -4.$$
$$4xy - 5y^2 = 3.$$

Multiplying the first equation by 3, the second by 4, and adding,

$$6x^2 + 7xy - 20y^2 = 0$$

or $\quad (3x - 4y)(2x + 5y) = 0.$

Equating each of these factors to zero, and solving with one of the given equations, we have, from the first factor,

$$(x_1, y_1) = (4, 3);\ (x_2, y_2) = (-4, -3);$$

from the second factor,

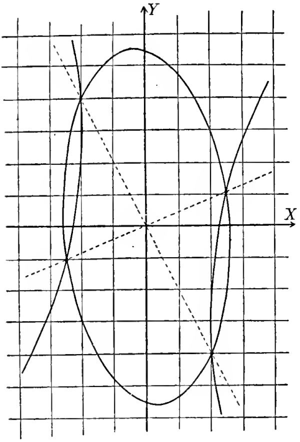
$$(x_3, y_3) = \left(\tfrac{1}{2}\sqrt{-5},\ -\tfrac{1}{5}\sqrt{-5}\right);\ (x_4, y_4) = \left(-\tfrac{1}{2}\sqrt{-5},\ \tfrac{1}{5}\sqrt{-5}\right).$$

Hence two solutions are real and two imaginary. The figure shows the graphs of the given equations and of the factors of the auxiliary equation.

To solve by the second method, transpose the constant term as before, then put $y = vx$.

We obtain $2x^2 - 3vx^2 = -4$; $4vx^2 - 5v^2x^2 = 3$.

Dividing, $\dfrac{2x^2 - 3vx^2}{4vx^2 - 5v^2x^2} = -\dfrac{4}{3}$ or $\dfrac{2 - 3v}{4v - 5v^2} = -\dfrac{4}{3}$.



Ellipse, $9x^2 + xy + 2y^2 = 60$
Hyperbola, $8x^2 - 3xy - y^2 = 40$
Straight lines, $(2x + y)(3x - 7y) = 0$

Clearing, etc., $20v^2 - 7v - 6 = 0$.

Hence, $v = \frac{3}{4}$ or $-\frac{2}{5}$.

Therefore $y = \frac{3}{4}x$ or $y = -\frac{2}{5}x$.

(These are the linear factors of the auxiliary equation found above.)

Substituting these values of y in either of the given equations, we find x as before.

Example 2.

$$9x^2 + xy + 2y^2 = 60,$$
$$8x^2 - 3xy - y^2 = 40.$$

The auxiliary equation is

$$6x^2 - 11xy - 7y^2 = 0,$$

or $(2x + y)(3x - 7y) = 0$.

Solving each factor with one of the given equations we obtain

$(x_1, y_1) = (2, -4)$; $(x_2, y_2) = (-2, 4)$;

$$(x_3, y_3) = \left(\frac{7}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}\right);$$

$$(x_4, y_4) = \left(-\frac{7}{2\sqrt{2}}, -\frac{3}{2\sqrt{2}}\right).$$

The graphs are given in the figure.

Exercises. Solve for x and y:

1. $\begin{cases} x^2 + y^2 = 9, \\ x^2 - xy = 10. \end{cases}$

2. $\begin{cases} x^2 - y^2 = 1, \\ x^2 - xy + y^2 = 1. \end{cases}$

3. $\begin{cases} 4x^2 - 9y^2 = 36, \\ y^2 + xy = -4. \end{cases}$

4. $\begin{cases} x^2 + 2xy = 2, \\ 2xy - y^2 = 6. \end{cases}$

5. $\begin{cases} x^2 + xy + y^2 = 3, \\ 2x^2 - 3y^2 = 6. \end{cases}$

6. $\begin{cases} 2x^2 + xy - 3y^2 = 2, \\ x^2 - xy + 2y^2 = 1. \end{cases}$

107. Case 3. The given equations are of the forms

$$ax^2 + by^2 = c,$$
$$a'x^2 + b'y^2 = c'$$

Rule. Consider x^2 and y^2 as the unknowns, and solve by the method of linear equations.

Graphically, we have two conics in standard form. The four solutions may all be real, or equal or imaginary in pairs.

Example. $x^2 - 4y^2 = 4,$
$9x^2 + 16y^2 = 144.$

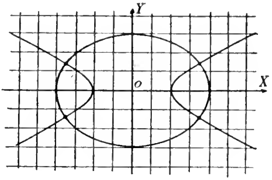
By elimination we obtain,

$$x^2 = \tfrac{160}{13};\quad y^2 = \tfrac{27}{13}.$$

Hence $x = \pm 4\sqrt{\tfrac{10}{13}};$

$$y = \pm 3\sqrt{\tfrac{3}{13}}.$$

Taking either value of x with either value of y, we obtain the four solutions. The approximate values may be scaled off from the Figure.



Hyperbola, $x^2 - 4y^2 = 4$
Ellipse, $9x^2 + 16y^2 = 144$

Exercises. Solve for x and y, and check graphically:

1. $\begin{cases} x^2 + y^2 = 4, \\ x^2 - y^2 = 2. \end{cases}$

2. $\begin{cases} x^2 - y^2 = 1, \\ x^2 + 4y^2 = 4. \end{cases}$

3. $\begin{cases} 2x^2 + 5y^2 = 10, \\ 4x^2 + y^2 = 4. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 9, \\ 4x^2 + 9y^2 = 36. \end{cases}$

5. $\begin{cases} 4x^2 + 5y^2 = 20, \\ x^2 - y^2 = 9. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 1, \\ x^2 + y^2 = 4. \end{cases}$

108. Case 4. Symmetric and Skew-Symmetric Equations. — A *symmetric equation* is one which remains unchanged when the variables are interchanged.

A *skew-symmetric equation* is one whose variable terms all change sign when the variables are interchanged. Thus

$$x^3 + y^3 + x + y = 0, \qquad x^3 - y^3 + 2x - 2y = 1$$

are symmetric and skew-symmetric respectively.

Rule. Given two such equations, put

$$x = u + v \quad \text{and} \quad y = u - v;$$

solve the resulting equations for u and v; then

$$x = \tfrac{1}{2}(u + v) \quad \text{and} \quad y = \tfrac{1}{2}(u - v).$$

Note. Equations of higher degree than the second may often be solved by this method.

Example.
$$x^4 + y^4 - x^2y^2 = 9,$$
$$x^2 + y^2 - xy = 3.$$

Let $x = u + v$ and $y = u - v.$

Substituting and reducing:

$$u^4 + 14\,u^2v^2 + v^4 = 9,$$
$$u^2 + 3\,v^2 = 3.$$

Let $u^2 = s$ and $v^2 = t.$

Then
$$s^2 + 14\,st + t^2 = 9,$$
$$s + 3\,t = 3.$$

Solving: $(s, t) = (3, 0)$ or $(\frac{3}{4}, \frac{3}{4})$.

(If s and t be considered as the coördinates of a point, the equations in s and t represent an ellipse and a straight line respectively.)

Since $u = \pm\sqrt{s}$ and $v = \pm\sqrt{t}$,

we have $(u, v) = (\pm\sqrt{3}, 0)$ or $\left(\pm\frac{\sqrt{3}}{2},\ \pm\frac{\sqrt{3}}{2}\right)$,

where the signs are to be taken in all possible ways.
Then

$$x = u + v = \sqrt{3},\ -\sqrt{3},\ \sqrt{3},\ -\sqrt{3},\quad 0,\quad 0;$$
$$y = u - v = \sqrt{3},\ -\sqrt{3},\quad 0,\quad 0,\ \sqrt{3},\ -\sqrt{3}.$$

Here corresponding values of x and y appear in the same vertical line.

109. Case 5. Symmetric Solution. — This method of solution is applicable to certain forms of symmetric equations, and may be illustrated by some simple examples.

Example 1.
$$x + y = 5,$$
$$xy = 4.$$

Squaring the first equation: $x^2 + 2\,xy + y^2 = 25.$

Subtracting four times the second: $x^2 - 2\,xy + y^2 = 9.$

Hence $x - y = \pm 3.$

But $x + y = 5.$

$\therefore\quad x = 4$ or 1; $y = 1$ or 4.

Example 2.
(1) $x^2 + xy + y^2 = 6.$
(2) $x^2 - xy + y^2 = 10.$

Subtract (2) from (1): $2\,xy = -4$, or $xy = -2.$

Add $xy = -2$ to (1): $x^2 + 2\,xy + y^2 = 4$, or $x + y = \pm 2.$

Subtract $3\,xy = -6$ from (1): $x^2 - 2\,xy + y^2 = 12$, or $x - y = \pm 2\sqrt{3}.$

Hence $x = \pm 1 \pm \sqrt{3}$ and $y = \pm 1 \mp \sqrt{3}.$

Simultaneous values of x and y are then obtained by taking the same combination of signs in these two results.

110. Miscellaneous methods for solving two simultaneous equations.

These methods depend on reducing the given equations, which may be of higher degree than the second, to one of the cases already discussed.

1. **By Substitution.** — This method has already been illustrated in several cases; in (**106**) we made the substitution $y = vx$, in (**107**) we put $x = u + v$ and $y = u - v$, and in example 2 of (**107**) we put $u^2 = s$ and $v^2 = t$. We shall give two more simple illustrations.

Example 1.
$$\frac{1}{x} + \frac{1}{y} = 2,$$
$$\frac{1}{xy} = -15.$$

If we let $\frac{1}{x} = s$ and $\frac{1}{y} = t$, and we obtain,
$$s + t = 2,$$
$$st = -15.$$

These may be solved by the method of (**109**).

Example 2.
$$x^2 + y^2 + x^2y^2 + 2xy = 4,$$
$$x^2y^2 - 2xy = 0.$$

Let $x + y = s$ and $xy = t$. Then
$$s^2 + t^2 = 4,$$
$$t^2 - 2t = 0.$$

The last two equations are readily solved, and give
$$s = +2;\ -2;\ 0.$$
$$t = \ \ 0;\ \ \ 0;\ 2.$$

The values of x and y may now be found by solving the pairs of equations,
$$\begin{cases} x + y = 2, \\ xy = 0. \end{cases} \quad \begin{cases} x + y = -2, \\ xy = 0. \end{cases} \quad \begin{cases} x + y = 0, \\ xy = 2. \end{cases}$$

2. **By modifying or combining the given equations** so as to obtain simpler forms. In particular, a common factor may sometimes be removed by division.

Example 1.

(1) $$x^2 - xy = 18y,$$
(2) $$xy - y^2 = 2x.$$

Dividing (1) by (2), we have
$$\frac{x}{y} = 9\frac{y}{x} \quad \text{or} \quad \left(\frac{x}{y}\right)^2 = 9 \quad \text{or} \quad x = \pm 3y.$$

Substituting each of these values of x in either of the given equations, we can solve for y and so complete the solution.

Example 2.

$$\begin{matrix}(1)\\(2)\end{matrix}\qquad \begin{cases} x^2y - x = 1, \\ x^4y^2 - x^2 = 3; \end{cases} \text{ or } \begin{cases} x(xy - 1) = 1, \\ x^2(x^2y^2 - 1) = 3. \end{cases}$$

Divide (2) by (1): $\qquad x(xy + 1) = 3.$

Divide this equation by (1): $\qquad \dfrac{xy + 1}{xy - 1} = 3.$

Hence $\qquad xy = 2.$

Then from (2), $\qquad x^2(4 - 1) = 3,$ or $x^2 = 1,$ or $x = \pm 1.$

But from (1), $\qquad x(2 - 1) = 1,$ or $x = 1.$

In this case the value $x = -1$ must be discarded.

Hence the only solution is $x = 1,\ y = 2.$

Example 3.

$$\begin{matrix}(1)\\(2)\end{matrix}\qquad \begin{aligned} x^4 + y^4 &= 1, \\ x - y &= 1. \end{aligned}$$

Raise (2) to the fourth power and subtract from (1):

(3) $\qquad 4x^3y - 6x^2y^2 + 4xy^3 = 0.$

Square (2) and multiply the result by $4xy$:

(4) $\qquad 4x^3y - 8x^2y^2 + 4xy^3 = 4xy.$

Subtract (4) from (3):

$$2x^2y^2 = -4xy, \quad \text{or} \quad x^2y^2 + 2xy = 0.$$

Hence $\qquad xy = 0,$ or $xy = -2.$

Solving each of the last two equations with (2) we have

$$(x, y) = (1, 0), (0, -1), \left(\frac{1 + \sqrt{-7}}{2}, \frac{-1 + \sqrt{-7}}{2}\right), \left(\frac{1 - \sqrt{-7}}{2}, \frac{-1 - \sqrt{-7}}{2}\right).$$

All four solutions also satisfy equation (1).

111. Summary of Methods for Solving Simultaneous Equations. — [Let the given equations be numbered (1) and (2).]

(a) Equation (1) linear, (2) quadratic.

Rule: Substitute from (1) in (2). *Graph,* straight line and conic.

(b) Equations (1) and (2) both quadratic.

Case 1. *Equation* (1) *is factorable.*

Rule: Put each factor separately equal to zero and solve with (2) as in (a). *Graph,* two straight lines and a conic.

Rule for factoring: Solve for y in terms of x (or x in terms of y); the quantity under the radical must be a perfect square.

Case 2. (1) $Ax^2 + Bxy + Cy^2 = D$; (2) $A'x^2 + B'xy + C'y^2 = D'$. Form the auxiliary equation, (1) $\times D'$ − (2) $\times D = 0$. Factor this and solve as in Case 1.

Second Method: Put $y = vx$ in (1) and (2) and divide results. *Graph,* two conics, centers at origin (except in case of parabola.)

Case 3. (1) $Ax^2 + By^2 = C$; (2) $A'x^2 + B'y^2 = C'$.

Solve as linear equations for x^2 and y^2.

Graph, two conics in standard position.

Case 4. *Symmetric Equations.*

Put $x = u + v$ and $y = u - v$.

Applicable to equations of higher degree.

Case 5. *Symmetric Solution* of certain symmetric equations.

(c) *Miscellaneous Methods.*

Exercises.

1. $x^2 + y^2 = 661.$ $x^2 - y^2 = 589.$

2. $y^2 - x^2 = -80.$ $x^2 + y^2 = 82.$

3. $3x^2 - y^2 = 59.$ $2x^2 + 3y^2 = 98.$

4. $x + y = 12.$ $xy = 35.$

5. $x + y = 1.$ $xy = -1.$

6. $x^2 + y^2 = 74.$ $x + y = 12.$

7. $x + y = \frac{7}{6}.$ $xy = \frac{1}{3}.$

8. $\dfrac{x + y}{x - y} = 12.$ $x^2 - y^2 = 48.$

9. $5x^2 + 2y^2 = 373.$ $2x + 5y = 54.$

10. $x^2 + y^2 = 10.$ $x - y = 2.$

11. $x^2 - y^2 = 120.$ $x + y = 20.$

12. $x^2 - y^2 = -\frac{1}{4}.$ $x + y = \frac{3}{2}.$

13. $x^2 + xy = 260.$ $xy + y^2 = 140.$

14. $x^2 + y^2 = 218.$ $xy - y^2 = 42.$

15. $5x + 2y = 29.$ $5xy = -105.$

16. $xy = 80.$ $x = 5y.$

17. $4x^2 - 3y^2 = -83.$ $3x + 2y = 26.$

18. $3x^2 - y^2 = 83.$ $x + y = 15.$

19. $xy + x = 20.$ $xy - y = 12.$

20. $2x + 3y = 20.$ $3xy - y^2 = 38.$

21. $5x^2 - 4y^2 = 109.$ $7x - 5y = 25.$

22. $x + xy + y = 47.$ $x + y = 12.$

23. $x^2 + xy + y^2 = 217.$ $x + y = 17.$

24. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2}.$ $\dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{5}{36}.$

25. $2x^2 - 3xy + y^2 = 3.$ $x^2 + 2xy - 3y^2 = 5.$

26. $x^2 - xy + y^2 = 37.$ $x^2 - y^2 = 40.$

27. $x^2 + xy + y^2 = 4.$ $x^2 - xy + y^2 = 2.$

28. $x^2 + xy + y^2 = 7.$ $x + y + xy = 5.$

29. $x^2 + y^2 = 5(x + y).$ $xy = 3.$

30. $x^3 + y^3 = 9.$ $xy = 4.$

31. $x^2 - 4y^2 = 4.$ $x^2 - 2xy + 2x = 4y.$

32. $2x^2 - 2y^2 + 3xy = -x - 2y.$ $x^2 - 4y^2 - x + 2y = 0.$

33. $u^2 + v^2 + uv = 67.$
$u + v = 9.$

34. $p^2 + pq + q^2 = 79.$
$p^2 - pq + q^2 = 37.$

35. $r^2 + s^2 + rs = 25.$
$r + s = 5.$

36. $r^2 + s^2 - rs = 84.$
$r - s = 2.$

37. $u + v + u^2 + v^2 = 162.$
$u - v + u^2 - v^2 = -102.$

38. $p + q + p^2 + q^2 = 1\frac{5}{8}.$
$q - p + q^2 - p^2 = -1.$

39. $x^2 + y^2 + x + y = 18.$
$2xy = 12.$

40. $h^2 + k^2 - k + h = 32.$
$2hk = 30.$

41. $x^2 + y^2 + x + y = 168.$
$\sqrt{xy} = 6.$

42. $m^2 + n^2 - m + n = 2400.$
$\sqrt{mn} = 30.$

43. $9u^2 + v^2 + 3u + v = 3042.$
$\sqrt{16uv} = 48.$

44. $r^3 - s^3 = 1304.$
$r - s = 8.$

45. $p^4 + q^4 = 337.$
$p + q = 7.$

46. $x^4 - y^4 = 609.$
$x - y = 3.$

47. $u^4 + v^4 = 2657.$
$u + v = 11.$

48. $m^3 + n^3 = 152.$
$m^2 - mn + n^2 = 19.$

49. $p + q + \sqrt{p + q} = 20.$
$p^3 + q^3 = 1072.$

50. $x^3 + y^3 = 280.$
$x^2 - xy + y^2 = 28.$

51. $u^2 + 3v^2 = 7.$
$7u^2 - 5uv = 18.$

52. $p^3 + q^3 = 152.$
$p^2q + pq^2 = 120.$

53. $x^3 - y^3 = 335.$
$xy^2 - x^2y = -70.$

54. $s^3 + t^3 = 855.$
$st(s + t) = 840.$

55. $m^3 - n^3 = 602.$
$mn(n - m) = -198.$

56. $u^2v^4 + v^2 = 17.$
$uv^2 + v = 5.$

57. $x^{\frac{3}{5}} + y^{\frac{1}{2}} = 35.$
$x^{\frac{1}{5}} + y^{\frac{1}{6}} = 5.$

58. $x^2y^2 - 18xy + 72 = 0.$
$6x^2 - 17xy + 12y^2 = 0.$

59. $x^4 + x^2y^2 + y^4 = 91.$
$x^2 - xy + y^2 = 7.$

60. $x^3 - y^3 = 7(x^2 - y^2).$
$x^2 + y^2 = 10(x + y).$

61. $s^6 + t^6 = 65.$
$s^4 + t^4 = 17.$

62. $x^2 + y^2 = a.$
$x^2 - y^2 = b.$

63. $x - y = m.$
$xy = n^2.$

64. $p^2 + q^2 = a^2.$
$p + q = b.$

65. $\sqrt{u} + \sqrt{v} = a.$
$u + v = b^2.$

66. $x^2 + y^2 = a(x - y).$
$x^2 + y^2 = b(x + y).$

67. $ax - by = m.$
$a^3x^3 - b^3y^3 = nxy.$

68. $b(x + y) = a(x - y).$
$x^2 + y^2 = m^2.$

69. $x^4 + y^4 = -8.$
$x - y = 2.$

70. $p^4 + q^4 = -9.$
$p - q = 3.$

71. $u^4 + v^4 = 175.$
$u - v = 5.$

72. $r^2 + rs + s^2 = a.$
$r^3s + rs^3 = b.$

73. $\frac{1}{x} - \frac{1}{y} = \frac{1}{a}.$
$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{b^2}.$

74. $u^2 + uv = m.$
$v^2 + uv = n.$

75. $x^3 + xy^2 = p.$
$y^3 + x^2 y = q.$

76. $m^3 - n^3 = a(m - n).$
$m^3 + n^3 = b(m + n).$

77. $r^5 + s^5 = 3368.$
$r + s = 8.$

78. $x(x + y - z) = 1.$
$y(x + y - z) = 2.$
$z(x + y - z) = 3.$

79. $x + y + z = 2.$
$xy = 3.$
$xyz = 6.$

80. $xy = 8z.$
$xz = 18y.$
$yz = \frac{9}{2}x.$

81. $xy + x = 1.$
$yz + y = -1.$
$xz + z = 3.$

82. $x(x + y - z) = a.$
$y(x + y - z) = b.$
$z(x + y - z) = c.$

83. $x + y + z = p.$
$xy = q.$
$xyz = r.$

84. $x(x + y + z) = a^2.$
$y(x + y + z) = b^2.$
$z(x + y + z) = c^2.$

85. $(x + y)(x + z) = 4.$
$(x + y)(y + z) = 1.$
$(x + z)(y + z) = 16.$

86. $(x + y)(x + z) = a.$
$(x + y)(y + z) = b.$
$(x + z)(y + z) = c.$

87. $xy + x = a.$
$yz + y = b.$
$xz + z = c.$

88. $x^2 + y^2 = 17z.$
$3(x + y) = 5z.$
$x - y = z.$

89. $x^2 + y^2 + z^2 = \frac{181}{144}.$
$y^2 + x = \frac{59}{48}.$
$z^2 + x = \frac{11}{12}.$

Problems. (10 ft) see answers, page 289

1. The hypothenuse of a right triangle is $\sqrt{100}$ ft. long. Find the other sides, if their ratio is 3 : 4.

2. The product of two numbers is 735, and their quotient $\frac{5}{3}$. Find the numbers.

3. Find two factors of 1728 whose sum is 84.

4. The sum of two numbers is 34. Three times their product exceeds the sum of their squares by 284. What are the numbers?

5. The product of two numbers increased by the first is 180, increased by the second is 176. What are the numbers?

6. The product of two numbers times their sum is 1820, times their difference 546. What are the numbers?

7. The sum of the squares of two numbers plus the sum of the numbers is 686. The difference of the squares plus the difference of the numbers is 74. What are the numbers?

8. The diagonal of a rectangle is 89 ft. long. If each side were 3 ft. shorter, the diagonal would be 4 ft. shorter. Find the sides.

9. The diagonal of a rectangle is 65 ft. long. If the shorter side were decreased by 17 ft. and the longer increased by 7 ft., the diagonal would be unchanged. Find the sides.

10. The diagonal of a rectangle is 85 ft. long. If each side be increased 2 ft. in length, the area is increased by 230 sq. ft. Find the sides.

11. The floor area of two square rooms is 890 sq. ft., and one room is 4 ft. larger each way than the other. Find the dimensions of each room.

12. For 60 yards of cloth B pays two dollars more than A pays for 45 yards. B receives one yard more for two dollars than does A. How much does each pay per yard?

13. Two bodies moving around the circumference of a circle of length 1260 ft. pass each other every 157.5 seconds. The first body makes the circuit in 10 seconds less than the second. Find the speed of each body.

14. The amount of a capital plus interest for one year is $22,781. If the capital were $200 larger and the rate of interest $\frac{1}{4}$% larger, the amount in one year would be $23,045. Find the capital and rate of interest.

15. A and B agree to do a piece of work in 6 days for $45. To finish on time, they hire C during the last two days, and consequently B gets $2 less pay. If A could have done the work alone in 12 days, how long would it take B and C, each working alone, to do it?

16. The quotient of a number of two digits divided by the product of the digits is 3. When the digits are interchanged, the new number is $\frac{7}{4}$ of the original. What is the number?

17. If the digits of a two-figure number be interchanged, the number is diminished by 18. The product of the original and the new number is 1008. What is the original number?

18. What number of two digits is 5 greater than twice the product of its digits and 4 less than the sum of their squares?

19. A fraction is doubled by adding 6 to its numerator and taking 2 from its denominator. If the numerator be increased and the denominator decreased by 3, the fraction is changed to its reciprocal. What is the fraction?

20. A and B start at the same time from two points 221 miles apart and travel towards each other. A goes 10 miles a day. B goes as many miles a day as the number of days until they meet diminished by 6. How far did each one travel?

21. The fore wheel of a wagon makes 1000 revolutions more than the hind wheel in going a distance of 7500 yards. Had the circumference of each wheel been one yard more, the difference between the number of revolutions would have been 625. Find the circumference of each wheel.

22. Find two numbers such that their sum shall be equal to 28, and the sum of their cubes divided by the sum of their squares equal to 1456.

23. Two points, A on the x-axis 270 ft. from the origin and B on the y-axis 189 ft. from the origin, move toward the origin. After 10 seconds the distance between them is 169 ft., and after 14 seconds, 109 ft. Find the speed of each point.

113. Exponential Equations. — An exponential equation is one in which the unknown appears in the exponent. Thus:

$$\sqrt{a^x} = a^{2x-1};\ (m^{x+1})^x = m^{-2x-2};\ a^{x+1} = b^{2x-1}.$$

Exponential equations of the above forms may be solved by reduction to ordinary equations by use of the principle that

if $$a^u = a^v, \text{ then } u = v,$$

or more generally,

if $$a^u = b^v, \text{ then } u \log a = v \log b.$$

Example 1. $$\sqrt{a^x} = a^{2x-1}.$$

This may be written $a^{\frac{x}{2}} = a^{2x-1}$.

$$\therefore \quad \frac{x}{2} = 2x - 1 \quad \text{or} \quad x = \frac{2}{3}.$$

Example 2. $$(m^{x+1})^x = m^{-2x-2}.$$

$$m^{x^2+x} = m^{-2x-2}.$$

$$\therefore \quad x^2 + x = -2x - 2 \quad \text{or} \quad x^2 + 3x + 2 = 0.$$

Hence $$x = -2 \text{ or } -1.$$

Example 3. $$2^{x+1} = 3^{2x-1}.$$

Taking logarithms: $$(x + 1) \log 2 = (2x - 1) \log 3.$$

$$\therefore \quad x (\log 2 - 2 \log 3) = - \log 2 - \log 3.$$

or $$x = \frac{\log 2 + \log 3}{2 \log 3 - \log 2} = \frac{\log 6}{\log \frac{9}{2}}.$$

Using common logarithms to four places,

$$x = \frac{0.7781 +}{0.6532 +} = 1.1912 +.$$

114. Exercises. Solve:

1. $2^x = 8$.
2. $2^x = \frac{1}{8}$.
3. $4^x = \frac{1}{64}$.
4. $(\frac{1}{4})^x = 2^{12}$.
5. $(\frac{1}{2})^x = 1$.
6. $(\frac{1}{125})^x = 25^3$.
7. $10^{-x} = 1000$.
8. $1000^x = 100$.
9. $3^{x+2} = 3^3$.
10. $\sqrt[5]{a^{x-2}} = a^{x-2}$.
11. $\sqrt[5]{p^{2x+8}} = p^0$.
12. $4^{2x-1} = 2^{6x+8}$.
13. $\sqrt[3]{m^x} = \sqrt{m^{3x+2}}$.
14. $\frac{1}{a^{5x}} = \sqrt[3]{a^{6-13x}}$.
15. $\sqrt[2x+1]{a^4} = \sqrt[3x-1]{a^5}$.
16. $3^x = \sqrt[3]{5}$.
17. $3^{x+1} = 5^{2x}$.
18. $4^{2x-3} = 7^{x-1}$.
19. $a^{3x+2} = b^{2x-1}$.
20. $3^{x^2-x-6} = 1$.
21. $8^{x^2+2x} = 512$.
22. $5^{\frac{x^3-8}{x-2}} = 25^{\frac{x^2-1}{2(x+1)}}$.
23. $(a^{x-2})^{3x+4} = a^{x(3x+\frac{1}{2})}$.
24. $(b^{x+3})^{3x-1} = b^{3x(x+1)}$.
25. $\sqrt[3]{e^{2-x}}\ \sqrt[4]{e^{4-x}}\ \sqrt[6]{e^{5x-1}} = 1$.

CHAPTER VI

Ratio, Proportion, Variation

115. *Definitions.* The **ratio** of two quantities is their indicated quotient.

Thus the ratio of a to b is $\frac{a}{b}$, or as it is usually written, $a : b$.

The numerator of the fraction, or the first term of the ratio, is called the **antecedent**, the other term the **consequent**.

The ratio $b : a$ is called the **inverse** of the ratio $a : b$.

Two ratios are equal when the fractions representing them are equal.

Since $$\frac{a}{b} = \frac{ma}{mb}, \quad \therefore\ a : b = ma : mb.$$

Hence, *both terms of a ratio may be multiplied by the same (or equal) quantities without altering the value of the ratio.*

Similarly, if $m \neq n$, then $a : b \neq ma : nb$.

Hence, *if the terms of a ratio be multiplied by unequal quantities, the value of the ratio is changed.*

The *compound ratio* of $a : b$ and $c : d$ is $ac : bd$, that is, the ratio of the product of the antecedents to the product of the consequents.

In particular the compound ratio of $a : b$ and $a : b$, or $a^2 : b^2$, is called the *duplicate ratio* of a to b; $a^3 : b^3$ is called the *triplicate ratio* of a to b, and so on.

A *proportion* is an equality of two ratios. Four numbers are in proportion when the ratio of two of them equals the ratio of the other two.

Four numbers a, b, c, d are in proportion if $a : b = c : d$ (often written $a : b :: c : d$). Here a and d are called the *extremes* and b and c the *means*. Also, d is called a *fourth proportional* to a, b, c.

The numbers $a, b, c, d, e, \ldots$ are in continued proportion if

$$a : b = b : c = c : d = d : e \cdots.$$

When three numbers a, b, c are in continued proportion, so that $a : b = b : c$, then b is called *a mean proportional* between a and c.

Since $\frac{a}{b} = \frac{b}{c}$ or $ac = b^2$ we have $b = \pm \sqrt{ac}$. Also, c is called the third proportional to a and b.

116. Laws of Proportion.

1. In a proportion, the product of the means equals the product of the extremes.

2. If two products, each containing two factors, are equal, either pair of factors may be taken as the means, the other as the extremes of a proportion.

When four numbers are in proportion so that $a : b = c : d$, then they are in proportion

3. by inversion, or $b : a = d : c$;
4. by alternation, or $a : c = b : d$;
5. by composition, or $a + b : b = c + d : d$

$$\left(\text{if } \frac{a}{b} = \frac{c}{d}, \quad \text{then } \frac{a}{b} + 1 = \frac{c}{d} + 1, \text{ or } \frac{a+b}{b} = \frac{c+d}{d}\right);$$

6. by division, or $a - b : b = c - d : d$;
7. by composition and division, or $a + b : a - b = c + d : c - d$.
8. Like powers (or roots) of the terms of a proportion are in proportion, i.e.,

$$\text{if } a : b = c : d, \text{ then } a^n : b^n = c^n : d^n.$$

$$\left(\text{For if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a^n}{b^n} = \frac{c^n}{d^n}.\right)$$

9. The products of the corresponding terms of any number of proportions are in proportion, i.e., if

$$a : b = c : d,\ a' : b' = c' : d',\ a'' : b'' = c'' : d'', \text{ etc.,}$$

then $$aa'a'' \cdots : bb'b'' \cdots = cc'c'' \cdots : dd'd'' \cdots.$$

$$\left(\text{For if } \frac{a}{b} = \frac{c}{d}, \frac{a'}{b'} = \frac{c'}{d'}, \frac{a''}{b''} = \frac{c''}{d''} \cdots, \text{ then } \frac{aa'a'' \ldots}{bb'b'' \ldots} = \frac{cc'c'' \ldots}{dd'd'' \ldots}.\right)$$

10. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent, i.e.,

$$a_1 : a_2 = b_1 : b_2 = c_1 : c_2 \ldots$$
$$= a_1 + b_1 + c_1 + \cdots : a_2 + b_2 + c_2 + \cdots.$$

For if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \cdots = r$, then $a_1 = a_2 r,\ b_1 = b_2 r,\ c_1 = c_2 r, \ldots$.
Hence $(a_1 + b_1 + c_1 + \cdots) = r\,(a_2 + b_2 + c_2 + \cdots)$, or

$$\frac{a_1 + b_1 + c_1 + \cdots}{a_2 + b_2 + c_2 + \cdots} = r.$$

11. More generally, if the ratios $a_1 : a_2,\ b_1 : b_2,\ c_1 : c_2 \ldots$ are all equal to each other, then

(a) $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \cdots = \frac{pa_1 + qb_1 + rc_1 + \cdots}{pa_2 + qb_2 + rc_2 + \cdots},$$

where $p, q, r, \ldots$ are any multipliers;

(b) $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \cdots = \sqrt[n]{\frac{a_1^n + b_1^n + c_1^n + \cdots}{a_2^n + b_2^n + c_2^n + \cdots}}.$$

Exercise. Prove 11 (a) and 11 (b). For what values of $p, q, r, \ldots n$ will these reduce to 10?

117. *Example.* Solve for x: $\frac{x + a}{x - a} = \frac{c}{d}$.

By composition and division: $\frac{2x}{2a} = \frac{c + d}{c - d}$.

$$\therefore \quad x = a\frac{c + d}{c - d}.$$

Exercises. Solve for x, using the laws of proportion:

1. $\frac{x + 1}{x} = \frac{3}{2}$.

2. $\frac{x}{x - 2} = -\frac{5}{6}$.

3. $\frac{2x - 3}{2x + 3} = \frac{7}{6}$.

4. $3x - 2 : 3x + 2 = 3 : 4$.

5. $\frac{x + 1}{x - 1} = \frac{x + 3}{x - 4}$.

6. $\frac{x + a}{x} = \frac{b}{c}$.

7. $a - x : x = p : q$.

8. $x + m : a = x - m : b$.

9. $a - x : x - b = a : b$.

10. $\frac{x + p}{x - p} = \frac{a + x}{b + x}$.

118. Variation. — A *variable* quantity is one which may be considered to assume a number of values.

A *function* of a variable is a quantity whose value depends on that of the variable.

If y be a function of a variable x [indicated by writing $y = f(x)$], then in general, as x varies y varies with it.

Thus, the circumference of a circle depends on the radius and varies with the radius. Hence the circumference is a function of the radius [$c = f(r)$]. The functional relation is expressed by $c = 2\pi r$.

Similarly, the area of a circle depends on the radius [$A = f(r)$]. The functional relation in this case is $A = \pi r^2$.

Also, the cost of a piece of cloth depends on, or is a function of, the price per yard; the running time of a train between two stations is a function of the speed; the range of a gun is a function of the muzzle velocity.

119. Direct Variation. — *A quantity y varies directly with another quantity x when their ratio remains constant.*

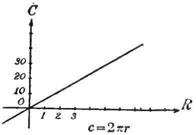
This is indicated by writing $y \propto x$ (read "y varies directly as x").

If k denote the constant value of the ratio of y to x, then **$y \propto x$ is exactly equivalent to $y = kx$.**

The constant k will be determined as soon as the value of y corresponding to a single value of x (other than $x = 0$) is known.

Graphically, the relation between y and x is represented by a straight line through the origin, the inclination of the line to the x-axis increasing with the absolute value of k. The line is completely determined by the origin ($x = 0$, $y = 0$) and one other point.

If c be the circumference and r the radius of a circle, then $c \propto r$, for $c = 2\pi r$. If we take $\pi = \frac{22}{7}$, then $c = \frac{44}{7}$ when $r = 1$. The figure gives the graph.



Horizontal scale = 10 times vertical scale

Exercise. From the figure read off to the nearest unit the lengths of circumference of circles whose radii are .15 in., .33 ft., 1.27 mm., .87 cm. respectively.

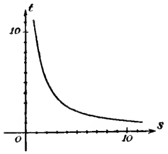
120. Inverse Variation. — When y varies directly as $\frac{1}{x}$, that is, $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$, then y is said to *vary inversely as x.*

When y varies inversely as x, this may be expressed by writing $xy = k$.

Graphically, the relation between x and y is then represented by a rectangular hyperbola, whose asymptotes are the coördinate axes.

If t be the time, in hours, required by a train to run 10 miles, and s the speed in miles per hour, then

$$t = \frac{10}{s} \quad \text{or} \quad t \propto \frac{1}{s}.$$



$st = 10$

The figure gives the graph, only positive values being considered.

Exercise 1. From the figure read off to tenths of a unit the times required to run 10 miles when $s = 4.5$, 7.8, and 15.6 miles per hour respectively.

Exercise 2. Construct a curve showing the possible dimensions of a rectangle whose area must be 16 sq. ft. Show that either dimension varies inversely as the other.

121. Joint Variation. — When a quantity varies directly as the product of two others, it is said to *vary with them jointly*.

Thus, if $z \propto xy$, or $z = kxy$, then z varies jointly as x and y.

122. Exercises.

1. Show that the area of a rectangle varies jointly as its dimensions.

2. Show that the volume of a right cylinder varies jointly as its base and altitude.

3. Same as in 2 for a right circular cone.

4. Show that the volume of a sphere varies jointly as the radius and the area of a great circle.

5. If $y \propto x$ and $x \propto z$, show that $y \propto z$.

6. If $x \propto z$ and $y \propto z$, show that $ax + by \propto z$.

7. If $x^2 \propto y$ and $z^2 \propto y$, show that $xz \propto y$.

8. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, show that $x \propto z$.

9. If x varies jointly as p and q, and y varies directly as $\frac{p}{q}$, show that p^2 varies jointly as x and y.

10. According to Boyle's law of gases, pressure times volume is constant. Show that the pressure (p) varies inversely as the volume (v). Show graphically the relation between p and v if $v = 1$ cu. ft. when $p = 25$ lbs. per sq. in.

11. If $w = uv$, show that $w \propto u$ when v is constant, and that $w \propto v$ when u is constant.

If $a : b = c : d$, show that

12. $4a + 5b : 3a + 2b = 4c + 5d : 3c + 2d$.

13. $a - 2b : -2a + b = c - 2d : -2c + d$.

14. $ma + nb : pa + qb = mc + nd : pc + qd$.

15. $3a + 2c : a - c = 3b + 2d : b - d$.

16. $\frac{1}{2}a - 4c : \frac{1}{2}b - 4d = 2a + \frac{1}{2}c : 2b + \frac{1}{2}d$.

17. $a : a + c = a + b : a + b + c + d$.

18. $a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2$.

19. $a + b : c + d :: \sqrt{a^2 + b^2} : \sqrt{c^2 + d^2}$.

20. $\sqrt{a^2 + b^2} : \sqrt{c^2 + d^2} = \sqrt[3]{a^3 + b^3} : \sqrt[3]{c^3 + d^3}$.

21. $\sqrt{a^2 + b^2} : \sqrt{c^2 + d^2} = \sqrt[3]{a^3 - b^3} : \sqrt[3]{c^3 - d^3}$.

If $a : b = c : d$ and $p : q = r : s$, show that

22. $p^m a^n : r^m b^n = q^m c^n : s^m d^n$.

23. $(a + b)(p - r) : (a - b)(p + r) = (c + d)(q - s) : (c - d)(q + s)$.

Solve for x:

24. $8ab : x = bc : 1\frac{3}{4}ac$.

25. $\frac{a + b}{a - b} : \frac{a^2 - b^2}{ab} = x : \frac{(a - b)^2}{ab}$.

26. $\left(\frac{a^3 - b^3}{a - b} + ab\right) : \left(\frac{a^3 + b^3}{a + b} - ab\right) = (a + b)^2 : x$.

27. The intensity of light varies inversely as the square of the distance from the source. If the sun is equivalent to 600,000 full moons in brightness, at how many times its present distance would it be of the same brightness as the full moon?

28. The squares of the periods of revolution of the planets about the sun vary as the cubes of their mean distances. The earth makes a revolution in one year at a mean distance of 93,000,000 miles. Venus makes a revolution in 225 days, Jupiter in 12 years. Find their mean distances from the sun.

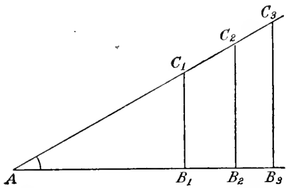
29. In beams of the same width and thickness the deflection due to a central load varies jointly as the load and the cube of the length. If a beam 10 ft. long is bent $\frac{1}{2}$ inch by a load of 1000 lbs., how much will a load of 5000 lbs. bend a 30-ft. beam?

30. Two lights, one of which is twice as strong as the other, are 10 ft. apart. Where on the line joining them do they produce equal illumination?

CHAPTER VII

The Trigonometric Functions

123. Consider any number of right triangles having a common acute angle A, as AB_1C_1, AB_2C_2, and AB_3C_3, in the figure.



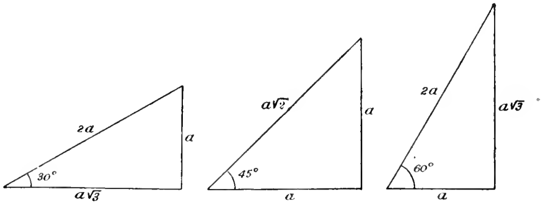
Since these triangles are similar, homologous sides are proportional, and therefore

$$\frac{B_1C_1}{AC_1} = \frac{B_2C_2}{AC_2} = \frac{B_3C_3}{AC_3} = \lambda,$$

λ (lambda) denoting the common value of the ratio of the side opposite $\angle A$ to the hypotenuse in the several triangles.

Evidently, in every right triangle having an acute angle equal to A the ratio of the side opposite $\angle A$ to the hypotenuse has the same value λ; λ depends only on $\angle A$, and not at all on the particular triangle in which this angle may be found. For example, if

$$A = 30°, \lambda = \tfrac{1}{2}; \quad A = 45°, \lambda = \frac{1}{\sqrt{2}}; \ A = 60°, \lambda = \tfrac{1}{2}\sqrt{3}.$$



Hence we see that λ is a function of A, and that to every value of A corresponds a definite value of λ.

This function is called the **sine of angle A**, or

$$\lambda = \textbf{sine of angle } A = \sin A.$$

124. The ratio of the side opposite the angle to the hypotenuse is merely one of six possible ratios which may be formed from the three sides of any right triangle. Hence associated with every acute angle there are six ratios, or six abstract numbers, whose values depend merely on the magnitude of the angle. They are called the **six trigonometric ratios**, or **trigonometric functions of the angle**, and are named as follows:

$$\text{sine of } \angle A = \sin A = \frac{\text{opposite side}}{\text{hypotenuse}}.$$

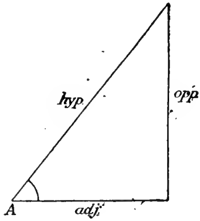
$$\text{cosine of } \angle A = \cos A = \frac{\text{adjacent side}}{\text{hypotenuse}}.$$

$$\text{tangent of } \angle A = \tan A = \frac{\text{opposite side}}{\text{adjacent side}}.$$

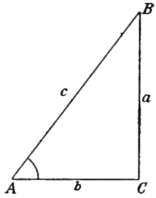
$$\text{cosecant of } \angle A = \csc A = \frac{\text{hypotenuse}}{\text{opposite side}}.$$

$$\text{secant of } \angle A = \sec A = \frac{\text{hypotenuse}}{\text{adjacent side}}.$$

$$\text{cotangent of } \angle A = \cot A = \frac{\text{adjacent side}}{\text{opposite side}}.$$



If the sides of the triangle are a, b, c, as in the figure, then



$$\sin A = \frac{a}{c}, \quad \csc A = \frac{c}{a},$$

$$\cos A = \frac{b}{c}, \quad \sec A = \frac{c}{b},$$

$$\tan A = \frac{a}{b}, \quad \cot A = \frac{b}{a}.$$

125. Exercises. With the aid of a protractor (see inside of back cover), construct triangles containing the following angles and, by measuring the sides and dividing, calculate to two decimals the six functions of these angles.

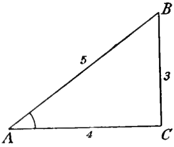
1. 30°.	**4.** 75°.	**7.** 85°.	**10.** 5°.
2. 45°.	**5.** 15°.	**8.** 80°.	**11.** 57°.
3. 60°.	**6.** 18°.	**9.** 10°.	**12.** 38°.

Check the results of the preceding exercises by means of the following table.

Angle.	Sin.	Cos.	Tan.	Cot.	Sec.	Csc.
0°						
5	0.087	0.996	0.087	11.430	1.004	11.474
10	0.174	0.985	0.176	5.671	1.015	5.759
15	0.259	0.966	0.268	3.732	1.035	3.864
20	0.342	0.940	0.364	2.748	1.064	2.924
25	0.423	0.907	0.466	2.144	1.103	2.366
30	0.500	0.866	0.577	1.732	1.155	2.000
35	0.574	0.819	0.700	1.428	1.221	1.743
40	0.643	0.766	0.839	1.192	1.305	1.556
45	0.707	0.707	1.000	1.000	1.414	1.414
50	0.766	0.643	1.192	0.839	1.556	1.305
55	0.819	0.574	1.428	0.700	1.743	1.221
60	0.866	0.500	1.732	0.577	2.000	1.155
65	0.906	0.423	2.145	0.466	2.366	1.103
70	0.940	0.342	2.748	0.364	2.924	1.064
75	0.966	0.259	3.732	0.268	3.864	1.035
80	0.985	0.174	5.671	0.176	5.759	1.015
85	0.996	0.087	11.430	0.087	11.474	1.004
90						

126. Given one function, to determine the other functions.— When a function of an acute angle is given, the angle may be constructed by writing the given function as a fraction, and constructing a right triangle, two of whose sides are the numerator and denominator of this fraction respectively, or equal multiples of these quantities. Also, since the third side of the triangle can be calculated from the other two, all the other functions of the angle may be found when one function is given.

Examples.



1. $$\tan A = \frac{3}{4}\left(= \frac{\text{opp. side}}{\text{adj. side}}\right).$$

Lay off $AC = 4$ and $CB = 3, \perp AC$.

Then $$AB = \sqrt{4^2 + 3^2} = 5.$$

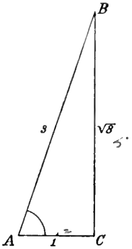
Hence $$\sin A = \frac{3}{5};\ \cos A = \frac{4}{5};$$

$$\csc A = \frac{5}{3};\ \sec A = \frac{5}{4};\ \cot A = \frac{4}{3}.$$

Scaling off the angle with a protractor, we have $A = 37°$. By taking from the table the angle whose tangent is .75 we have $A = 37°$ as before.

2. $$\text{Sec } A = 3\left(= \frac{3}{1} = \frac{\text{hyp.}}{\text{adj. side}}\right).$$

Lay off $AC = 1$. With A as center and radius $= 3$, strike an arc to cut the $\perp$ drawn to AC at C. This determines the point B.



The solution may now be completed as in example 1.

Another method of constructing the triangle in this example is to calculate CB first, and then to proceed as in example 1.

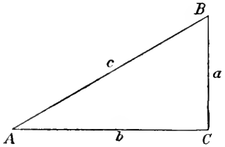
127. Exercises. Determine the angle (approximately) and the remaining functions, when

1. $\sin A = \frac{5}{13}$.
2. $\sin A = \frac{2}{3}$.
3. $\sin A = 0.6$.
4. $\cos A = \frac{2}{3}$.
5. $\cos A = \frac{1}{2}$.
6. $\tan A = \frac{3}{2}$.
7. $\tan A = 4$.
8. $\tan A = \sqrt{3}$.
9. $\cot A = 1$.
10. $\cot A = 1.5$.
11. $\sec A = 2$.
12. $\csc A = \frac{3}{2}$.
13. $\cos A = 0.2$.
14. $\csc A = 2.4$.
15. $\tan A = 10$.

16. Show that the equation $\sin A = 2$ is impossible.
17. Show that the equation $\cos A = 1.1$ is impossible.
18. Show that the equation $\sec A = \frac{1}{2}$ is impossible.
19. Show that the equation $\csc A = .9$ is impossible.

When A is an acute angle show that,

20. $\sin A$ lies between 0 and 1.
21. $\cos A$ lies between 0 and 1.
22. $\sec A$ and $\csc A$ are always greater than 1.
23. $\tan A$ and $\cot A$ may have any value from 0 to ∞.



128. Functions of Complementary Angles. — Since the sum of the two acute angles of a right triangle is 90°, they are complementary.

By definition, we have

$$\sin B = \frac{\text{opp. side}}{\text{hyp.}} = \frac{b}{c} = \cos A.$$

Also, $\cos B = \sin A$; $\tan B = \cot A$; $\csc B = \sec A$; $\sec B = \csc A$; and $\cot B = \tan A$.

Complementary Functions, or Co-functions. — The co-sine is called the complementary function to the sine and conversely. Similarly tangent and co-tangent are mutually complementary, and secant and co-secant.

The preceding equations are now all contained in the following

Rule: *Any function of an acute angle is equal to the co-function of the complementary angle.*

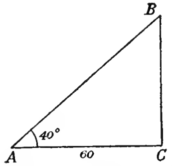
Exercise. Verify this rule when $A = 30°$, $45°$, and $60°$.

129. Application of the Trigonometric Functions to the Solution of Right Triangles. — When two parts of a right triangle are known, exclusive of the right angle, the triangle may be constructed and the remaining parts determined graphically. By the aid of tables of the trigonometric functions, the unknown parts may also be calculated.

Rule: When two parts of a right triangle are given (the rt. ∠ excepted) and a third part is required, write down that equation of (**124**) which involves the two given parts and the required part. Substitute in it the values of the given parts, and solve for the required part.

An exceptional case arises when two sides are given and the third side is required. In this case we may use the formula $a^2 + b^2 = c^2$. It will usually be better however, unless the given sides are represented by small numbers, to solve for one of the angles first, and then to obtain the third side from this angle and one of the given sides.

Example 1. In △ ABC, given $A = 40°$, $C = 90°$, and $b = 60°$. Find the other parts of the triangle.



To get B, we have $B = 90° - A = 50°$.

To get a, take $\frac{a}{b} = \tan A$ or $a = b \tan A$.

Finally, c is determined from $\frac{b}{c} = \cos A$

or $c = \frac{b}{\cos A} = b \sec A$.

From the table of (**125**), $\tan 40° = 0.839$ and $\sec 40° = 1.305$.

Hence $a = 60 \times 0.839 = 50.340$ and $c = 60 \times 1.305 = 78.300$.

As a check, we should have

$$\frac{a}{c} = \cos B \quad \text{or} \quad \frac{50.340}{78.300} = 0.643.$$

130. Exercises.

Determine the unknown parts of right triangle ABC, C being 90°, from the parts given below. Check results by graphic solution and by a check formula containing the unknown parts. (Use the table of (**125**).)

1. $A = 25°$, $a = 100$.

2. $A = 70°$, $b = 150$.

3. $A = 51°$, $c = 75$.

4. $B = 38°$, $c = 50$.

5. $B = 65°$, $b = 750$.

6. $B = 10°$, $a = 0.15$.

7. $A = 4'$, $c = 0.045$.

8. $B = 85°$, $c = 1.25$.

9. $B = 57°$, $a = 16\frac{2}{3}$.

10. $A = 20°$, $b = \frac{1}{25}$.

11. Find the length of chord subtended by a central angle of 100° in a circle of radius 50 ft. (First find the half-chord.)

12. Find the central angle subtended by a chord of 80 ft. in a circle of radius 200 ft.

13. Find the radius of the circle in which a chord of 100 ft. subtends an angle of 70°.

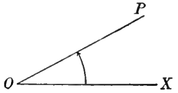
14. Find the length of side of a regular pentagon inscribed in a circle of radius 500 ft.

15. Find the length of side of a regular decagon circumscribed about a circle of radius 100 ft.

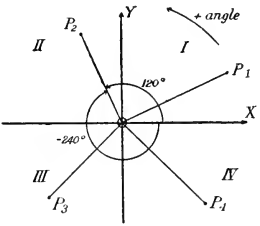
16. From a point in the same horizontal plane as the foot of a flag-pole, and 300 ft. from it, the angle of elevation of the top is 22°. How high is the pole?

17. A vertical pole 75 ft. high casts a shadow 60 ft. long on level ground. Find the altitude of the sun.

131. Angles of any Magnitude, Positive or Negative. — Consider $\angle XOP$ (figure) as generated by a moving line which rotates about O from the position OX to the position OP.



Divide the plane into four quadrants (I, II, III, and IV in the figure below) by means of two rectangular axes OX and OY.



In the figure, the arrows on the axes indicate the positive directions, and quadrant I is that included between the positive parts of the axes. Let a moving line start from the position OX and rotate into the positions OP_1, OP_2, OP_3, and OP_4 successively, thus generating the angles XOP_1, XOP_2, XOP_3, and XOP_4 respectively.

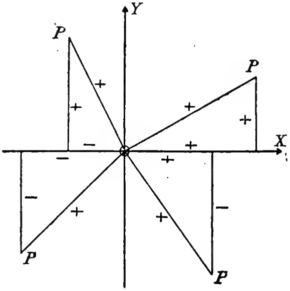
OX is called the **initial line**, and OP_1 the **terminal line** of the angle XOP_1, and similarly for any other angle.

An angle is **positive** when the generating line rotates **counter-clockwise** (in the direction of the curved arrow in the figure) **negative** when the generating line moves **clockwise.**

The **quadrant of an angle** is that quadrant in which its terminal line lies. The angle is said to lie in this quadrant.

The initial line OX, and any terminal line, as OP_2, may always be considered to form two angles numerically $< 360°$, as $+120°$ and $-240°$ in the figure.

When the moving line rotates from OX through more than one complete revolution, an angle greater than 360° is generated. Thus a rotation in the positive direction (positive rotation) through $1\frac{1}{3}$ revolutions generates an angle of 480°, lying in the second quadrant; a negative rotation through $2\frac{1}{6}$ revolutions generates an angle of $-780°$, lying in the fourth quadrant.



132. The Trigonometric Functions of any Angle.—Let XOP be any angle, and P any point in its terminal line. (The four possible cases are here shown in the figure, according to the quadrant of the angle.) Let OM be the abscissa of P, MP (not PM) the ordinate of P, and OP the distance of P. The signs of these quantities are taken according to the usual convention and are shown in the figure. We now define the functions of angle XOP, in whatever quadrant it may be, as follows:

$$\sin XOP = \frac{\text{ordinate (of } P)}{\text{distance (of } P)}; \qquad \csc XOP = \frac{\text{distance}}{\text{ordinate}};$$

$$\cos XOP = \frac{\text{abscissa}}{\text{distance}}; \qquad \sec XOP = \frac{\text{distance}}{\text{abscissa}};$$

$$\tan XOP = \frac{\text{ordinate}}{\text{abscissa}}; \qquad \cot XOP = \frac{\text{abscissa}}{\text{ordinate}}.$$

When $\angle XOP < 90°$, these definitions agree with those of (**124**).

According to the above definitions we have the following

TABLE OF SIGNS OF THE TRIGONOMETRIC FUNCTIONS

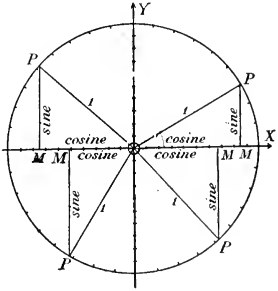
Quadr.	sin.	cos.	tan.	cot.	sec.	csc.
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

Let the student verify carefully the signs in this table. He should be prepared to state instantly the sign of any function in any quadrant.

Observe that in the first quadrant all the functions are positive; in the other quadrants a function and its reciprocal are positive, the remaining four negative.

133. Approximate Values of the Functions of any Angle. — If in the last figure the distances OP had been taken all of the same length, all the points P would lie on the circumference of a circle with center at O.

Let us draw a circle with O as center and unit radius (figure; $1 = 10$ small divisions). Then for any angle XOP we have



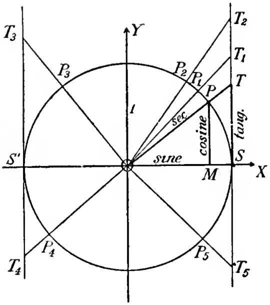
$$\sin XOP = MP \left(= \frac{MP}{1}\right),$$

$$\cos XOP = OM \left(= \frac{OM}{1}\right).$$

Hence approximate values of the sines and cosines of all angles may be read off directly from the figure. The other functions may be obtained by division, since $\tan XOP = \frac{MP}{OM}$, etc. They may also be constructed graphically by a method explained in the next article.

The lines OM and MP, whose lengths represent the sine and cosine of $\angle XOP$, are commonly referred to as the **line values** of these functions.

134. Line Values of the other Trigonometric Functions. — Construct a circle as in the figure below and draw the tangents at S and S'. Let XOP be an angle in the first quadrant. Produce OP to meet the tangent at S in T. Then by similar triangles,



$$\tan XOP = \frac{MP}{OM} = \frac{ST}{OS} = ST.$$

In the same way,

$$\tan XOP_1 = ST_1;$$
$$\tan XOP_2 = ST_2.$$

Hence when an angle is in the first quadrant, its tangent is measured by the segment of the tangent line from S to the terminal line produced; the radius of the circle is the unit of length.

By taking into account the algebraic sign of the tangent, we find that

$$\tan XOP_3 = -S'T_3;\ \tan XOP_4 = -S'T_4;\ \tan XOP_5 = ST_5.$$

Here $S'T_4$ and ST_5 are themselves negative lines.

Hence the *numerical* value of the tangent of any angle equals the segment of the vertical tangent cut off by the terminal line produced, this segment being measured in terms of the radius as unity. This value should be given the proper sign according to the quadrant of the angle.

We have further,

$$\sec XOP = \frac{OP}{OM} = \frac{OT}{OS} = OT.$$

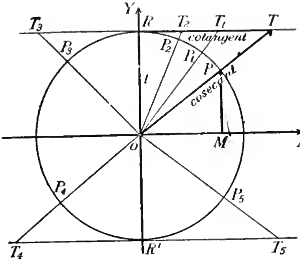
By examining the other angles shown in the figure we see that the numerical value of the secant of any angle equals the segment of the terminal line produced from the origin to the vertical tangent. The proper sign may be determined according to the quadrant of the angle.

To obtain graphic constructions of the cotangent and cosecant, we draw the tangents at R and R' (figure below). Then

$$\cot XOP = \frac{OM}{MP} = \frac{RT}{OR} = RT;$$

$$\csc XOP = \frac{OP}{MP} = \frac{OT}{OR} = OT.$$

By examining the other angles in the figure we see that, (a) the cotangent of any angle is numerically equal to the length of the segment of the horizontal tangent cut off by the terminal line of the angle produced; (b) the cosecant is numerically equal to the segment of the terminal line produced from the origin to the horizontal tangent.



In either case the sign is to be determined according to the quadrant of the angle.

135. Variation of the Trigonometric Functions. — In the figure of (**133**) suppose the point P to describe the circumference of the circle in such a way that the angle XOP shall vary continuously from 0° to 360°. Let us trace the changes in the value of $\sin XOP = MP$. In the first quadrant MP, and hence $\sin XOP$, varies from 0 to $+1$, in the second from $+1$ to 0, in the third from 0 to -1 and in the fourth from -1 to 0.

Similarly $\cos XOP$ varies in the four quadrants successively from $+1$ to 0, 0 to -1, -1 to 0, and 0 to $+1$.

Consider next $\tan XOP = \frac{MP}{OM}$. When $XOP = 0°$, $MP = 0$ and $OM = 1$; hence $\tan 0° = 0$.

Now as XOP increases from 0° toward 90°, MP steadily increases toward 1, while OM steadily diminishes toward 0. Hence $\tan XOP$ increases from 0 without limit, so that we write $\tan 90° = \infty$, and say that the tangent varies from 0 to ∞ as XOP varies from 0° to 90°.

Since the three remaining functions are reciprocals of the three already considered, their variations are easily traced. Thus, $\csc XOP = \frac{1}{\sin XOP}$. Hence $\csc XOP$ varies from ∞ to 1 in the first quadrant, and from 1 to ∞ in the second. Now as XOP passes through 180°, $\csc XOP$ changes suddenly from a large positive value when the angle is a little less than 180° to a large negative value when the angle is a little more than 180°.

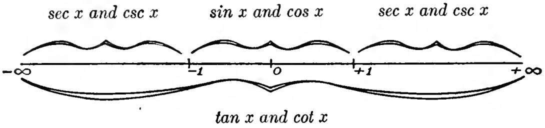
This abrupt change in the cosecant when the angle passes through 180° is expressed by saying that the cosecant has a **discontinuity** at 180°; sec 180° may be either $+\infty$ or $-\infty$, according to the side from which XOP approaches 180°.

In the third quadrant csc XOP is negative and varies from $-\infty$ to -1; in the fourth quadrant from -1 to $-\infty$. There is another discontinuity at 360° or 0°.

The variations of the six functions are shown in the following table.

Quadr.	sin.	csc.	cos.	sec.	tan.	cot.
I	0 to $+1$	$+\infty$ to $+1$	$+1$ to 0	$+1$ to $+\infty$	0 to $+\infty$	$+\infty$ to 0
II	$+1$ to 0	$+1$ to $+\infty$	0 to -1	$-\infty$ to -1	$-\infty$ to 0	0 to $-\infty$
III	0 to -1	$-\infty$ to -1	-1 to 0	-1 to $-\infty$	0 to $+\infty$	∞ to 0
IV	-1 to 0	-1 to $-\infty$	0 to $+1$	$+\infty$ to 1	$-\infty$ to 0	0 to $-\infty$

The range of values covered by each of the six functions is indicated in the diagram below.



Exercises.

1. Trace carefully the variations given in the above table.

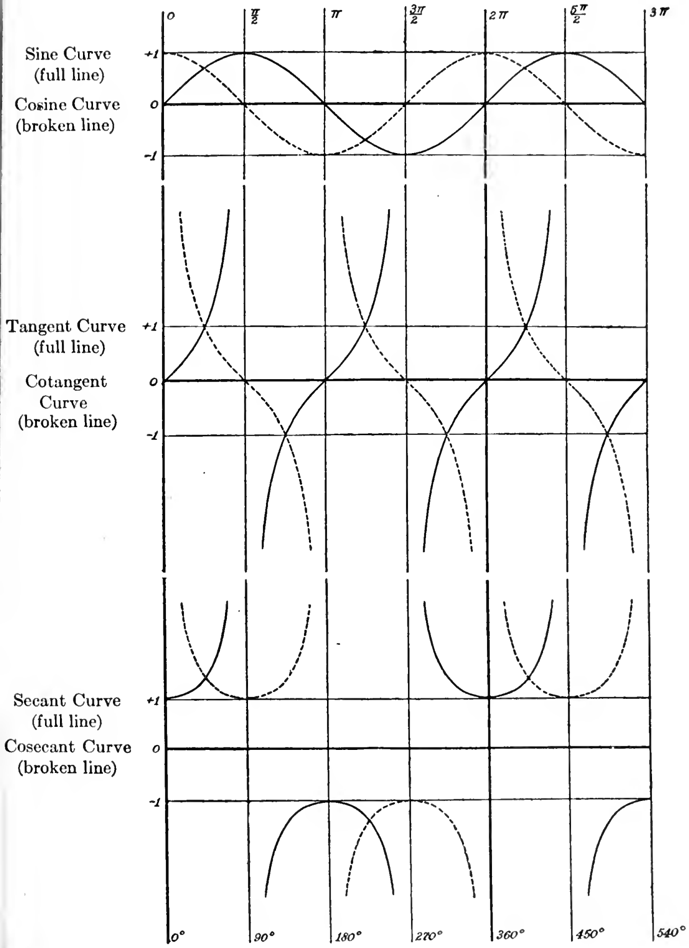
2. Show that the following functions have discontinuities at the values stated: the tangent, at 90° and 270°; the cotangent, at 0° and 180°; the secant, at 90° and 270°; the cosecant at 0° and 180°.

3. Discuss the "equations," $\tan 90° = +\infty$; $\tan 90° = -\infty$. Same for $\csc 0° = +\infty$; $\csc 0° = -\infty$.

4. Draw a circle as in the figure of (**133**), adding also the vertical and horizontal tangents. Divide the circumference into 36 equal parts, and obtain from the diagram a two-place table of the six functions for every tenth degree from 0° to 360°.

5. By use of the results of exercise 4, trace the graph of the equation $y = \sin x$. [Take angle x on a horizontal scale, making one small square = 10°. On the vertical scale choose any convenient length as 1 (= sin 90°), say 10 small squares. At successive points x on the horizontal axis erect perpendiculars equal to $\sin x$, upward or downward according to the sign. See note at end of (**143**)].

Graphs of the Trigonometric Functions



6. Trace the graph of $y = \cos x$. (On same diagram as $y = \sin x$.)
7. Trace the graphs of $y = \tan x$ and $y = \cot x$.
8. Trace the graphs of $y = \sec x$ and $y = \csc x$.

136. Periodicity of the Trigonometric Functions. — Since the position of the terminal line of an angle x is unchanged when the angle is increased or diminished by integral multiples of 360°, any function of x equals the same function of $x \pm n.360°$, n being an integer. That is,

$$f(x) = f(x \pm n.360°),$$

where f stands for any one of the trigonometric functions.

Hence the trigonometric functions are *periodic*, with a *period of* 360°. (See graphs on p. 105.)

137. Relations between the Functions of an Angle. — From the general definitions of the functions given in (**133**) we have, putting $\angle XOP = x$,

$$\sin x = \frac{1}{\csc x}; \quad \cos x = \frac{1}{\sec x}; \quad \tan x = \frac{1}{\cot x}.$$

$$\tan x = \frac{\text{ordinate}}{\text{abscissa}} = \frac{\frac{\text{ordinate}}{\text{distance}}}{\frac{\text{abscissa}}{\text{distance}}} = \frac{\sin x}{\cos x}; \quad \cot x = \frac{\cos x}{\sin x}.$$

Whatever be the quadrant of angle $XOP = x$ [figure of (**132**)], we have

$$(\text{ordinate})^2 + (\text{abscissa})^2 = (\text{distance})^2.$$

Dividing this equation through in turn by $(\text{distance})^2$, $(\text{abscissa})^2$, and $(\text{ordinate})^2$, and expressing the resulting ratios as functions we have

$$\sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x,$$
$$1 + \cot^2 x = \csc^2 x.$$

All the above relations between the functions of an angle x are true for all values of x. They form a first set of working formulas, and should be thoroughly committed to memory. They are collected below, as

Formulas, Group A.

(1) $\sin x = \dfrac{1}{\csc x}$.

(2) $\cos x = \dfrac{1}{\sec x}$.

(3) $\tan x = \dfrac{1}{\cot x}$.

(4) $\tan x = \dfrac{\sin x}{\cos x}$.

(5) $\cot x = \dfrac{\cos x}{\sin x}$.

(6) $\sin^2 x + \cos^2 x = 1$.

(7) $1 + \tan^2 x = \sec^2 x$.

(8) $1 + \cot^2 x = \csc^2 x$.

We shall apply these formulas in two examples.

Example 1. Prove that $\tan x + \cot x = \sec x \csc x$.

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$$
$$= \frac{1}{\sin x \cos x} = \csc x \sec x.$$

Example 2. Prove that

$$\frac{\csc x}{\tan x + \cot x} = \cos x.$$

$$\frac{\csc}{\tan x + \cot x} = \frac{\csc x}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}} = \frac{\csc x}{\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}}$$
$$= \frac{\csc x}{\dfrac{1}{\sin x \cos x}} = \csc x \sin x \cos x = \cos x.$$

In both examples all the steps taken are true for all values of x, since this is true of all the formulas of group A. Hence the given equations are true for all values of x, and they are therefore called *trigonometric identities*.

The equation $\sin^2 x - \cos^2 x = 1$ is not true for all values of x, but holds only for certain special values; it is not an identity.

138. Exercises. Prove the following identities:

1. $\tan x \cos x = \sin x$.

2. $\dfrac{1}{\cot x \sec x} = \sin x$.

3. $\tan x = \dfrac{\sec x}{\csc x}$.

4. $\cot x = \dfrac{\csc x}{\sec x}$.

5. $(\sin^2 x + \cos^2 x)^2 = 1$.

6. $\dfrac{\cos\theta}{\sin\theta \tan\theta} = \cot^2\theta$.

7. $(\csc\theta - \cot\theta)(\csc\theta + \cot\theta) = 1$.

8. $(\sec x - \tan x)(\sec x + \tan x) = 1$.

9. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$.

10. $\sin^2\alpha + \cos^2\alpha = \sec^2\alpha - \tan^2\alpha$.

11. $(\sin\alpha - \cos\alpha)^2 = 1 - 2\sin\alpha\cos\alpha$.

12. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x.$

13. $(1 - \sin^2 x) \csc^2 x = \cot^2 x.$

14. $\cot^2 \theta - \cos^2 \theta = \cot^2 \theta \cos^2 \theta.$

15. $\tan \theta + \cot \theta = \sec \theta \csc \theta.$

16. $\tan \phi \sin \phi + \cos \phi = \sec \phi.$

17. $\sin^2 \phi \sec^2 \phi = \sec^2 \phi - 1.$

18. $\dfrac{\sin \phi}{1 - \cos \phi} = \dfrac{1 + \cos \phi}{\sin \phi}.$

19. $\dfrac{1 + \tan^2 \beta}{1 + \cot^2 \beta} = \dfrac{\sin^2 \beta}{\cos^2 \beta}.$

20. $(1 - \sin^2 \beta)(1 + \tan^2 \beta) = 1.$

21. $\tan^4 x - \sec^4 x = 1 - 2 \sec^2 x.$

22. $\dfrac{\cos x + \sin x}{\cos x - \sin x} = \dfrac{1 + \tan x}{1 - \tan x}.$

23. $(\tan x - 1)(\cot x - 1) = 2 - \sec x \csc x.$

24. $\sec \theta + \tan \theta = \dfrac{\cos \theta}{1 - \sin \theta}.$

25. $\sec \theta \sin^3 \theta = (1 + \cos \theta)(\tan \theta - \sin \theta).$

26. $\tan^2 \alpha + \cot^2 \alpha + 2 = \sec^2 \alpha \csc^2 \alpha.$

27. $\sin^3 \theta + \cos^3 \theta = (\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta).$

28. $(\sin^2 \theta - \cos^2 \theta)^2 = 1 - 4 \cos^2 \theta + 4 \cos^4 \theta.$

29. $\sin^6 \theta + \cos^6 \theta = \sin^4 \theta + \cos^4 \theta - \sin^2 \theta \cos^2 \theta.$

30. $(\sin x - \cos x)(\sec x - \csc x) = \sec x \csc x - 2.$

31. $\dfrac{\tan x - \cot x}{\tan x + \cot x} = \dfrac{2}{\csc^2 x} - 1.$

32. $(a \cos x - b \sin x)^2 + (a \sin x + b \cos \phi)^2 = a^2 + b^2.$

33. $\cos^2 \phi + (\sin \phi \cos \theta)^2 + (\sin \phi \sin \theta)^2 = 1.$

34. $\tan \alpha + \tan \beta = \tan \alpha \tan \beta (\cot \alpha + \cot \beta).$

139. Functions of any Angle in Terms of Functions of an Acute Angle. — It is possible to express in a simple manner any function of any angle in terms of a function of an acute angle. Therefore a table of values of the functions of angles from 0° to 90° will serve for all angles. In fact, in view of (**128**), a table of functions from 0° to 45° would be sufficient, though not convenient.

1. Any angle, positive or negative, can be brought into the first quadrant by adding to it, or subtracting from it, an integral multiple of 90°.

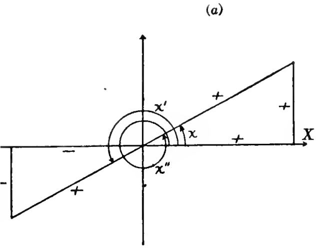
Thus: $760° - 8 \times 90° = 40°; \quad -470° + 6 \times 90° = 70°.$

2. When an angle is changed by an integral multiple of 90°, say $n \times 90°$, the new terminal line lies in the *same line* as the original terminal line when *n is even; at right angles* to it when *n is odd.*

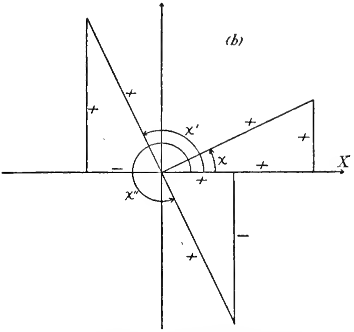
3. Two angles which differ by an *even multiple of* 90° will be called *symmetrical* with respect to the initial line, or simply *symmetrical;* two angles which differ by an *odd multiple of* 90°, *skew-symmetrical.*

4. When two angles are *symmetrical*, any function of the one is *numerically* equal to the *same function* of the other.

From figure (a), $\sin x = -\sin x' = \sin x''$, etc., for the other functions.



Angles x′ and x″ are symmetrical with respect to angle x



Angles x′ and x″ are skew-symmetrical with respect to x

When two angles are *skew-symmetrical*, any function of the one is *numerically* equal to the *co-function* of the other.

From figure (b), $\sin x = -\cos x' = \cos x''$, etc., for the other functions.

Exercise 1. From figures (a) and (b), write down all the functions of x in terms of functions of x' and of x''.

Exercise 2. Draw figures corresponding to figures (*a*) and (*b*), when x lies in each of the other quadrants. Then proceed as in exercise 1.

5. **Rule:** *Any function of any angle x is numerically equal to the* $\left\{\begin{array}{l}\textit{same function}\\ \textit{co-function}\end{array}\right.$ *of x increased or diminished by any* $\left\{\begin{array}{l}\textit{even}\\ \textit{odd}\end{array}\right.$ *multiple of* 90°.

As an equation,

$$f(x) = \begin{cases} \pm f(x \pm n \cdot 90^\circ), & n \text{ even;} \\ \pm \text{co-}f(x \pm n \cdot 90^\circ), & n \text{ odd.} \end{cases}$$

The sign of the result must be determined by noting the quadrants of x and $x \pm n \cdot 90°$.

When the new angle, $x \pm n \cdot 90°$, lies in the first quadrant, give to the result the sign of the given function of x, $f(x)$.

Examples.

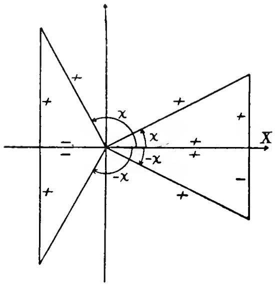
1. $\sin 680° = \sin (50° + 7 \times 90°) = -\cos 50°$.

Here we diminish the given angle by an odd multiple of 90°, hence change to the co-function. Also sin 680° is negative, hence we use the minus sign.

2. $\tan(-870°) = \tan(30° - 10 \times 90°) = +\tan 30°$.

3. $\sec 420° = \sec(60° + 4 \times 90°) = +\sec 60°$.

140. Relations between the Functions of $+x$ and $-x$.—The figure shows two cases, x in the first quadrant and x in the second quadrant. In either case,



$$\sin x = -\sin(-x);$$
$$\csc x = -\csc(-x);$$
$$\cos x = \cos(-x);$$
$$\sec x = \sec(-x);$$
$$\tan x = -\tan(-x);$$
$$\cot x = -\cot(-x).$$

Exercise. Show that these equations are true when x lies in the third quadrant or fourth quadrant.

Rule: *The cosine or secant of any angle is equal to the cosine or secant respectively of the negative angle; the remaining four functions of the angle are equal to the negative of the corresponding functions of the negative angle.* Or,

$f(x) = f(-x)$ *when f stands for cos. or sec.*

$f(x) = -f(-x)$ *when f stands for sin., csc., tan., or cot.*

141. Exercises. Express all the functions of the following angles in terms of functions of acute angles:

1. 130°.	**5.** 359°.	**9.** – 321°.	**13.** – 1060°.
2. 165°.	**6.** – 25°.	**10.** 742°.	**14.** – 401°.
3. 230°.	**7.** – 125°.	**11.** – 665°.	**15.** 525°.
4. 340°.	**8.** – 250°.	**12.** 1100°.	**16.** – 101°.

Express all the functions of the following angles in terms of functions of angles between 0° and 45°.

17. 75°.	**19.** 110°.	**21.** – 335°.	**23.** 790°.
18. – 80°.	**20.** 255°.	**22.** 600°.	**24.** – 510°.

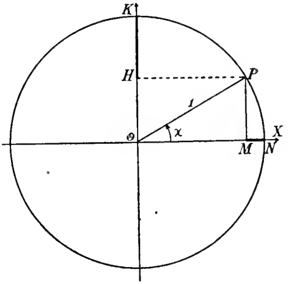
Find the values of the functions of:

25. 120°.	**29.** − 30°.	**33.** − 240°.
26. 135°.	**30.** − 45°.	**34.** 315°.
27. 150°.	**31.** − 60°.	**35.** 600°.
28. 300°.	**32.** − 120°.	**36.** − 510°.

142. Versed Sine and Coversed Sine. — The expressions $1 - \cos x$ and $1 - \sin x$ occur often enough in the applications of trigonometry to warrant the use of special symbols for them. These are

$$1 - \cos x \equiv \text{versed sine of } x \equiv \text{vers } x;$$
$$1 - \sin x \equiv \text{coversed sine of } x \equiv \text{covers } x.$$

Their line values are (figure), vers $x = MN$, covers $x = HK$, x being in the first quadrant.



Exercises. Find the values of the versed sine and coversed sine of:

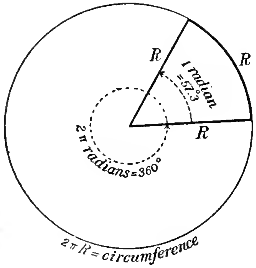
1. 30°.	**7.** 150°.
2. 45°.	**8.** − 30°.
3. 60°.	**9.** − 120°.
4. 90°.	**10.** − 225°.
5. 120°.	**11.** − 300°.
6. 135°.	**12.** − 315°.

143. Radian Measure. — The *degree* is an artificial unit for the measurement of angles. In France, where at the time of the Revolution an attempt was made to put all measurements on the basis of the decimal scale, the quadrant of the circle was divided into 100 equal parts and the angle subtended at the center by one part called a *grade*. Each grade was then subdivided into 100 equal parts called *minutes*, and each minute into 100 *seconds*. The degree and the grade are thus two arbitrary units for the measurement of angles, and any number of such units might be chosen.

There is one unit which is naturally related to the circle, and which is as commonly used in theory as the degree in practice. *It is the central angle subtended by an arc equal in length to the radius of the circle, and is called a* ***radian*** (figure, p. 112).

Since the circumference contains the radius 2π times, the entire central angle of 360° contains 2π radians, i.e.,

$$2\pi \text{ radians} = 360°.$$



Hence,

π radians $= 180°$;

$\frac{\pi}{2}$ radians $= 90°$;

$\frac{\pi}{4}$ radians $= 45°$; and so on.

In dealing with angles measured in radians it is customary to omit specifying the unit used; it is understood that when no unit is indicated the radian is implied. Thus, $2\pi = 360°$, $\pi = 180°$,

$$\frac{\pi}{3} = 60°,\ 2\tfrac{1}{2} = 2\tfrac{1}{2} \text{ radians, and so on.}$$

NOTE. To get the true form of the graphs of the equations $y = \sin x$, $y = \cos x$, etc., take x in radians on the x-axis, thus: $x = 0.1, 0.2, 0.3, \ldots, 1, \ldots$ and find the corresponding values of y; use the same unit of length for both x and y. See graphs on p. 105.

144. Radians into degrees, and conversely.

Since 2π (radians) $= 360°$,

therefore, $$\textbf{1 radian} = \frac{360°}{2\pi} = \frac{180°}{\pi} = \frac{180°}{3.1416-} = 57°.29+;$$

also, $$\textbf{1 degree} = \frac{2\pi}{360} \text{ (radians)} = \frac{\pi}{180} \text{ (radians)}$$
$$= \frac{1}{57.29+} \text{ (radians)} = .017+ \text{(radians)}.$$

Rule: *To convert radians into degrees, multiply the number of radians by* $\frac{180}{\pi}$ *or* $57.29+$.

To convert degrees into radians, multiply the number of degrees by $\frac{\pi}{180}$ *or* $\frac{1}{57.29+}$ *or* $.017+$.

By taking a sufficiently accurate value of π, we find,

1 radian $= 57°.2957795 = 3437'.74677 = 206264''.8$.

$1° = .0174533$ radians.

$1' = .0002909$ radians (point, 3 ciphers, 3, approx.).

$1'' = .0000048$ radians (point, 5 ciphers, 5, approx.).

The measure of an angle in radians is often called the *circular measure* of the angle.

145. Exercises. Reduce to degrees, minutes and seconds the angles whose circular measures are:

1. $\frac{\pi}{8}, \frac{3\pi}{2}, \frac{5\pi}{6}, \frac{5\pi}{8}, \frac{7\pi}{3}.$

2. $1, 2, \frac{1}{2}, \frac{1}{3}, \frac{7}{4}.$

3. $\frac{1}{12}\pi, -1\frac{1}{2}, \pi + 1, \frac{\pi}{2} + \frac{1}{3}, \frac{2\pi + 3}{6}.$

4. $\frac{1}{4} + \pi, \frac{\pi}{4} - \frac{1}{3}, \frac{1}{\pi}, \frac{2}{\pi - 3}, \pi^2.$

5. $\frac{\pi}{\pi^2 + 1}, \frac{\pi^2}{1 - \pi}, \frac{\pi + 1}{\pi - 1}.$

Reduce the following angles to circular measure:

6. 30°, 120°, 150°, 225°, −60°.

7. 375°, $-22\frac{1}{2}$°, 187°.5, 106°, 93° 45′.

8. 85°, 191° 15′, 5° 37′ 30″, 90° 37′ 30″.

9. 10′, 10″, 0″.1, 12° 5′ 4″, 21° 36′ 8″.1.

10. If the radius of the earth be taken as 3960 miles, find the number of feet in an arc of 1″ of the meridian.

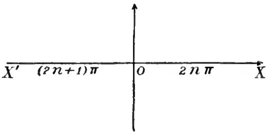
11. How many radians in a central angle subtended by an arc 75 ft. long, the radius of the circle being 50 ft.?

12. How many radians in the central angle subtended by the side of a regular inscribed decagon?

13. A wheel makes 1000 revolutions a minute. Find its angular velocity in radians per second.

14. If the angular velocity of a wheel is 10π radians per second, how many revolutions per minute does it make?

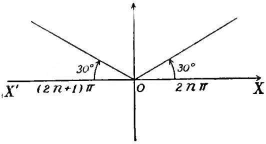
146. Angles Corresponding to a Given Function.—Let n denote an integer positive or negative, or zero; then $2n$ is always even, and $2n + 1$ odd; hence the angle $2n\pi$ has the terminal line OX (figure) coincident with the initial line, and angle $(2n + 1)\pi$ has the terminal line OX'.



Suppose now we wish to write down all angles x such that $\sin x = \frac{1}{2}$. Corresponding to a given function, there are always (except when the angle is a multiple of 90°) two angles less than 360°; in this case they are

$$30° \text{ and } \pi - 30°.$$

All angles with the same terminal line as either one of these will have the same functions; all such angles are



$$2\,n\pi + 30° \text{ and } 2\,n\pi + (\pi - 30°) = (2\,n + 1)\,\pi - 30°.$$

Hence all solutions of the equation $\sin x = \frac{1}{2}$ are given by

$$x = 2\,n\pi + 30° \text{ or } (2\,n + 1)\,\pi - 30°.$$

In general, if θ denote the smallest positive angle whose sine is a, then all solutions of the equation

(1) $\sin x = a$ are $x = 2\,n\pi + \theta$ and $(2\,n + 1)\,\pi - \theta$.

Hence also, if θ denote the smallest positive angle whose cosecant is a, the solutions of the equation

(2) $\csc x = a$ are $x = 2\,n\pi + \theta$ and $(2\,n + 1)\,\pi - \theta$.

Consider next the equation

$$\cos x = \tfrac{1}{2}.$$

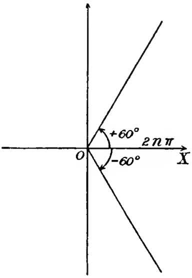
The two simplest solutions are

$$x = +60° \quad \text{and} \quad x = -60°.$$

All possible solutions are given by

$$x = 2\,n\pi + 60° \quad \text{and} \quad x = 2\,n\pi - 60°,$$

or $$x = 2\,n\pi \pm 60°.$$



In general, if θ be the smallest positive angle whose cosine is a, all solutions of the equation

(3) $\cos x = a$ are $x = 2\,n\pi \pm \theta$.

Hence also, if θ be the smallest angle whose secant is a, all solutions of the equation

(4) $\sec x = a$ are $x = 2\,n\pi \pm \theta$.

Finally consider the equation

$$\tan x = 1.$$

The two simplest solutions are

$$x = 45° \quad \text{and} \quad x = \pi + 45°,$$

and all possible solutions are

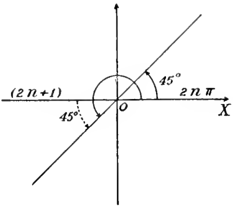
$$x = 2\,n\pi + 45° \quad \text{and} \quad x = 2\,n\pi + (\pi + 45°),$$

the second set being the same as $x = (2\,n + 1)\,\pi + 45°$.

Both sets are contained in the single equation

$$x = n\pi + 45°,$$

the first set being obtained when n is even, the second set when n is odd.



In general, if θ be the smallest positive angle whose tangent is a, all solutions of the equation

(5) $\tan x = a$ are $x = n\pi + \theta$.

Hence also, if θ be the smallest positive angle whose cotangent is a, all solutions of the equation

(6) $\cot x = a$ are $x = n\pi + \theta$.

Summary of equations (1) to (6).

Let θ denote the smallest positive angle having a given function equal to a given number a. *Then all solutions of the equation*

I. $\begin{cases} \sin x = a \\ \csc x = a \end{cases}$ are $x = 2\,n\pi + \theta$ and $(2\,n + 1)\,\pi - \theta$;

II. $\begin{cases} \cos x = a \\ \sec x = a \end{cases}$ are $x = 2\,n\pi \pm \theta$;

III. $\begin{cases} \tan x = a \\ \cot x = a \end{cases}$ are $x = n\pi + \theta$.

The angle θ is usually called the **principal value** of x.

The solutions of these equations may also be written by the following simple rule.

Rule: Corresponding to a given value of a function, there are in general two and only two positive angles less than 360°. If these be denoted by x_1 and x_2, then all possible angles are given by $x_1 \pm 2\,n\pi$ and $x_2 \pm 2\,n\pi$.

In exceptional cases there may be only one angle $< 360°$, as when $\sin x = 1$ or $\cos x = -1$.

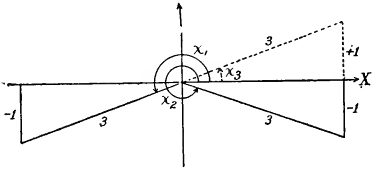
147. Use of Tables of Natural Functions. — Usually the angles corresponding to a given value of a function are not known exactly. The angles may then be found approximately by the aid of tables of the natural functions, such as are given in (**125**) and in Appendix, Table III.

These tables give the functions of angles from 0° to 90°. But they will serve for all four quadrants, since any function of any angle is reducible to a function of an acute angle.

When the given value of the function is not found exactly in the table, the corresponding angle must be obtained by *interpolation.*

Example 1. Given $\sin x = -\frac{1}{3}$. To find x.

The two values, x_1 and x_2, $< 360°$, are shown in the figure. They are



easily found when x_3, the angle whose sine is $+\frac{1}{3}$, is known. For

$$x_1 = \pi + x_3 \quad \text{and} \quad x_2 = 2\pi - x_3.$$

Since $\sin x_3 = \frac{1}{3} = .333$, we find by interpolation from Table III, $x_3 =$ 19° 28′. Hence, $x_1 = 199° 28'$, $x_2 = 340° 32'$.

All possible values of x are then given by

$$199° 28' \pm 2n\pi, \ 340° 32' \pm 2n\pi.$$

Example 2. Given $\cot \frac{2}{3} x = 3.362$. To find x.

From Table III, $\frac{2}{3} x = 16° 34'$ or $196° 34'$ ($= 180° + 16° 34'$).

Hence all possible values of $\frac{2}{3} x$ are given by

$$\frac{2}{3} x = 16° 34' \pm 2n\pi \quad \text{or} \quad 196° 34' \pm 2n\pi.$$

Therefore, $x = 24° 51' \pm 3n\pi$ or $294° 51' \pm 3n\pi$.

We might also write, from III of (**146**),

$$\frac{2}{3} x = 16° 34' + n\pi; \text{ hence } x = 24° 51' + \frac{3}{2} n\pi.$$

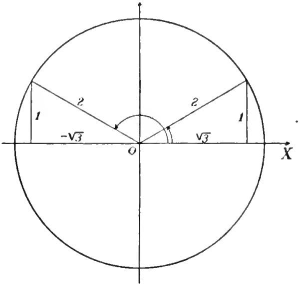
148. Exercises. Find all values of the angles which satisfy the following equations:

1. $\cot x = 1$; $\sin x = -\frac{1}{2}$; $\sec x = 2$; $\cos x = 1$.
2. $\csc x = -\sqrt{2}$; $\tan x = \sqrt{3}$; $\cos x = .5$; $\cot x = -\sqrt{3}$.
3. $\sin x = -\frac{3}{5}$; $\sec x = -3$; $\tan x = 2$; $\csc x = 5$.
4. $\cos x = -.257$; $\cot x = -.998$; $\sin x = .020$.
5. $\tan \theta = 2.500$; $\csc \theta = -3.505$; $\sec \theta = -10$.
6. $\text{vers}\, \phi = 1.450$; $\text{vers}\, \phi = .605$; $\text{covers}\, \phi = .750$.

149. Given one function of an angle, to find the other functions.

Example 1. $\sin x = \frac{1}{2}\cdot$ Find the other functions.

Take ordinate = 1 and distance = 2; then abscissa = $\pm\sqrt{3}$ (figure). Then

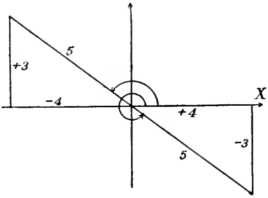


$$\cos x = \pm\frac{\sqrt{3}}{2},\quad \tan x = \pm\frac{1}{\sqrt{3}},$$

$$\cot x = \pm\sqrt{3},$$

$$\sec x = \pm\frac{2}{\sqrt{3}},\quad \csc x = 2.$$

We have found *two values* for each function except csc x, which is the reciprocal of the given function. Similar results will be found in general.



Example 2.

$$\tan x = -\frac{3}{4}\ \left(=\frac{-3}{+4}\ \text{or}\ \frac{+3}{-4}\right).$$

The two possible positions of the terminal line are shown in the figure.

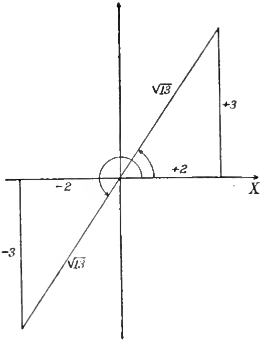
Hence, $\sin x = \pm\frac{3}{5}$, $\cos x = \pm\frac{4}{5}$,

$$\cot x = -\frac{4}{3},\ \csc x = \pm\frac{5}{3},\ \sec x = \pm\frac{5}{4}.$$

Example 3.

$$\cot x = \frac{2}{3}\ \left(=\frac{+2}{+3}\ \text{or}\ \frac{-2}{-3}\right)$$

Then (figure),



$$\sin x = \pm\frac{3}{\sqrt{13}},\quad \cos x = \pm\frac{2}{\sqrt{3}},$$

$$\tan x = \frac{3}{2},$$

$$\csc x = \pm\frac{\sqrt{13}}{3},\quad \sec x = \pm\frac{\sqrt{13}}{2}.$$

Example 4. $\sin x = \frac{h}{k}\cdot$

Ordinate = h; distance = k; hence abscissa = $\pm\sqrt{k^2 - h^2}$.

Then $$\cos x = \pm \frac{\sqrt{k^2 - h^2}}{k}, \quad \tan x = \pm \frac{h}{\sqrt{k^2 - h^2}}, \text{ etc.}$$

Exercise 1. Construct figures for the cases when $\frac{h}{k}$ is (a) plus; (b) minus.

Exercise 2. Is the problem possible for all values of h and k?

Example 5. $$\tan x = \frac{a - b}{2\sqrt{ab}} \quad \left(= \frac{-(a-b)}{-2\sqrt{ab}} \right).$$

Here $\quad$ ordinate $= a - b$, abscissa $= 2\sqrt{ab}$;

or, $\quad$ ordinate $= -(a - b)$, abscissa $= -2\sqrt{ab}$.

In either case, distance $= +\sqrt{(a-b)^2 + 4ab} = |a + b|$.

Hence, $$\sin x = \pm \frac{a - b}{|a + b|}, \quad \cos x = \pm \frac{2\sqrt{ab}}{|a + b|}, \text{ etc.}$$

Exercise 1. Calculate the values of the six functions when $a = 2$, $b = 3$; when $a = -2$, $b = -3$; when $a = 1$, $b = 4$; $a = -1$, $b = -4$.

Exercise 2. Is the problem possible for all values of a and b?

150. Exercises. Find the other functions, given that

1. $\sin x = -\frac{1}{2}$.
2. $\cos x = \frac{1}{2}$.
3. $\tan x = \frac{4}{3}$.
4. $\sec x = 4$.
5. $\cot x = \sqrt{3}$.
6. $\csc x = -\frac{13}{12}$.
7. $\sec x = -\frac{41}{40}$.
8. $\cot x = -.75$.
9. $\sin x = .6$.
10. $\cos \theta = \frac{b}{c}$.
11. $\csc \theta = -\frac{m}{n}$.
12. $\tan \theta = a$.
13. $\sin \phi = h$.
14. $\cot \phi = \sqrt{c}$.
15. $\sec \phi = \frac{a^2 + b^2}{2ab}$.
16. State for what values of the literal quantities in exercises 10–15, the given equations are impossible.

151. To express all the functions in terms of one of them.

1. Express all the functions in terms of the cosine.

We have
$$\cos x = \frac{\cos x}{1} = \frac{\text{abscissa}}{\text{distance}}.$$

Hence let $\quad$ abscissa $= \cos x$ and distance $= 1$.

Then $\quad$ ordinate $= \pm\sqrt{\text{dist.}^2 - \text{absc.}^2} = \pm\sqrt{1 - \cos^2 x}$.

The figure shows this graphically when $\cos x$ is positive. Taking into account both values of the ordinate, we have

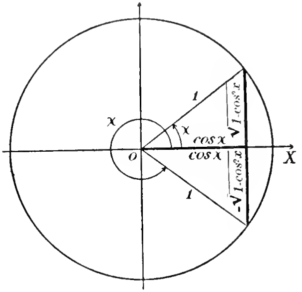
$$\sin x = \pm\sqrt{1-\cos^2 x};$$

$$\tan x = \pm\frac{\sqrt{1-\cos^2 x}}{\cos x};$$

$$\cot x = \pm\frac{\cos x}{\sqrt{1-\cos^2 x}};$$

$$\csc x = \pm\frac{1}{\sqrt{1-\cos^2 x}};$$

$$\sec x = \frac{1}{\cos x}.$$

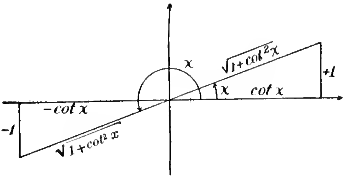


Exercise 1. Obtain these equations for the case when $\cos x$ is negative.

Exercise 2. Obtain the same equations directly from the formulas of Group A.

2. Express all the functions in terms of the cotangent.

$$\cot x = \frac{\cot x}{1} = \frac{-\cot x}{-1} = \frac{\text{abcsissa}}{\text{ordinate}}.$$



Hence let abscissa $= \cot x$ and ordinate $= 1$;

or let abscissa $= -\cot x$ and ordinate $= -1$.

In either case, distance $= +\sqrt{1+\cot^2 x}$. (See figure, where we assume $\cot x > 0$.)

Hence $\sin x = \pm\dfrac{1}{\sqrt{1+\cot^2 x}}$, $\cos x = \pm\dfrac{\cot x}{\sqrt{1+\cot^2 x}}$, etc.

By taking each of the functions in turn, and proceeding as above, we obtain the results shown in the following table. The given function and its reciprocal are uniquely determined; the other four functions are ambiguous in sign.

	sin x.	cos x.	tan x.	cot x.	sec x.	csc x.
sin x	. . . .	$\pm\sqrt{1-\cos^2 x}$	$\frac{\tan x}{\pm\sqrt{1+\tan^2 x}}$	$\frac{1}{\pm\sqrt{1+\cot^2 x}}$	$\frac{\pm\sqrt{\sec^2 x-1}}{\sec x}$	$\frac{1}{\csc x}$
cos x	$\pm\sqrt{1-\sin^2 x}$	. . . .	$\frac{1}{\pm\sqrt{1+\tan^2 x}}$	$\frac{\cot x}{\pm\sqrt{1+\cot^2 x}}$	$\frac{1}{\sec x}$	$\frac{\sqrt{\csc^2 x-1}}{\csc x}$
tan x	$\frac{\sin x}{\pm\sqrt{1-\sin^2 x}}$	$\frac{\pm\sqrt{1-\cos^2 x}}{\cos x}$	. . . .	$\frac{1}{\cot x}$	$\pm\sqrt{\sec^2 x-1}$	$\frac{1}{\pm\sqrt{\csc^2 x-1}}$
cot x	$\frac{\pm\sqrt{1-\sin^2 x}}{\sin x}$	$\frac{\cos x}{\pm\sqrt{1-\cos^2 x}}$	$\frac{1}{\tan x}$	. . . .	$\frac{1}{\pm\sqrt{\sec^2 x-1}}$	$\pm\sqrt{\csc^2 x-1}$
sec x	$\frac{1}{\pm\sqrt{1-\sin^2 x}}$	$\frac{1}{\cos x}$	$\pm\sqrt{1+\tan^2 x}$	$\frac{\pm\sqrt{1+\cot^2 x}}{\cot x}$	. . . .	$\frac{\csc x}{\pm\sqrt{\csc^2 x-1}}$
csc x	$\frac{1}{\sin x}$	$\frac{1}{\pm\sqrt{1-\cos^2 x}}$	$\frac{\pm\sqrt{1+\tan^2 x}}{\tan x}$	$\pm\sqrt{1+\cot^2 x}$	$\frac{\sec x}{\pm\sqrt{\sec^2 x-1}}$	. . . .

Exercises.

1. Express $\sin x \cos^2 x + \sin^3 x$ in terms of $\tan x$.

2. Express $\tan x \sec x + \sec^2 x$ in terms of $\sin x$.

3. Express $\cos^2 x \tan x + \sin^2 x \cot x$ in terms of $\csc x$.

4. Express $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta}$ in terms of $\sec\theta$.

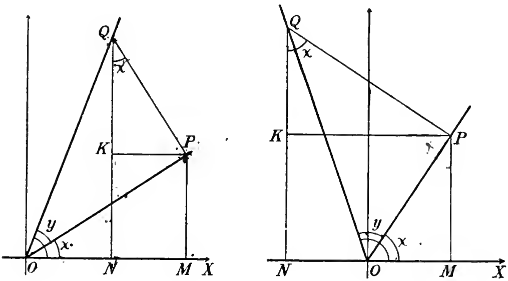
5. Express $\frac{\cos\theta}{1-\tan\theta} + \frac{\sin\theta}{1-\cot\theta}$ in terms of $\cos\theta$.

CHAPTER VIII

FUNCTIONS OF SEVERAL ANGLES

152. Formulas for sin $(x+y)$ and cos $(x+y)$. — Let x and y be two angles, each of which we first assume to be less than 90°. Their sum will then fall in the first or the second quadrant. The two cases are illustrated in the figures, and the demonstration which follows applies to either figure.

Construct $\angle XOP = x$ and $\angle POQ = y$, the terminal side of x being taken as the initial side of y.



From Q, any point on the terminal side of y, draw perpendiculars NQ and PQ to the sides of angle x, produced if necessary. Draw $MP \perp OX$ and $KP \perp NQ$.

Then $\angle KQP = x$, and in either figure,

$$\sin(x+y) = \frac{NQ}{OQ} = \frac{MP+KQ}{OQ} = \frac{MP}{OQ} + \frac{KQ}{OQ}$$

$$= \frac{MP}{OP}\cdot\frac{OP}{OQ} + \frac{KQ}{PQ}\cdot\frac{PQ}{OQ}.$$

Hence

(a) $\quad\quad$ **sin $(x+y)$ = sin x cos y + cos x sin y.**

Also, noting that ON in the second figure is a negative line,

$$\cos(x+y) = \frac{ON}{OQ} = \frac{OM - NM}{OQ} = \frac{OM}{OQ} - \frac{KP}{OQ}$$

$$= \frac{OM}{OP} \cdot \frac{OP}{OQ} - \frac{KP}{PQ} \cdot \frac{PQ}{OQ}.$$

Hence

(b) $$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

153. In the above proofs we have assumed x and y less than 90°. Similar proofs may be given for any other values of x and y.

We shall however use formulas (a) and (b) to verify the truth of the formulas

(a′) $$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$
(b′) $$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

for all values of A and B.

A and B will differ from acute angles by certain integral multiples of 90°, say,

$$A = x + n \cdot 90°;\ B = y + m \cdot 90°.$$

All possible quadrants for A and B (except the first, for which the formulas have been derived) will be included by considering only the values 1, 2, 3 for n and m.

In particular, let $n = 1$ and $m = 2$. Then

$$A = x + 90°;\ B = y + 180°;\ A + B = x + y + 270°.$$

Hence, if formulas (a′) and (b′) are true,

$$\sin(x + y + 270°) = \sin(x + 90°)\cos(y + 180°) + \cos(x + 90°)\sin(y + 180°),$$

$$\cos(x + y + 270°) = \cos(x + 90°)\cos(y + 180°) - \sin(x + 90°)\sin(y + 180°).$$

Removing the multiples of 90° by the rule of (**139**) and changing signs, these equations reduce to

$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

But these are true since x and y are acute angles; hence also (a′) and (b′) are true. In exactly the same way the truth of these equations may be shown for any integral values of n and m, positive or negative.

Using the letters x and y in place of A and B, formulas (a) and (b) are true for all values of x and y.

154. Replacing y by $-y$ in (a) and (b), and noting that

$\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$, we have

(c) $$\sin(x - y) = \sin x \cos y - \cos x \sin y;$$

(d) $$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$

Equations (a), (b), (c), (d) are usually called the addition and subtraction formulas of trigonometry. All the other working formulas are deduced from them.

155. Dividing (a) by (b), we have

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

$$= \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{1 - \dfrac{\sin x \sin y}{\cos x \cos y}}.$$

Hence,

(e) $$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Similarly,

(f) $$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}.$$

Also, from (c) and (d), by division,

(g) $$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

(h) $$\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}.$$

Exercises.

1. If $\sin x = \frac{1}{3}$ and $\sin y = \frac{2}{3}$, calculate $\sin(x + y)$.

(Four answers: $\frac{1}{9}[\pm\sqrt{5} \pm 4\sqrt{2}]$.)

2. If $\cos x = \frac{4}{5}$ and $\cos y = \frac{40}{41}$, calculate $\cos(x + y)$.
3. If $\sin \alpha = \frac{4}{5}$ and $\sin \beta = \frac{3}{5}$, calculate $\cos(\alpha - \beta)$.

Show that,

4. $\cos(60° + x) + \cos(60° - x) = \cos x$.
5. $\sin(45° + \theta) - \sin(45° - \theta) = \sqrt{2}\sin\theta$.
6. $\cot\theta + \tan\phi = \dfrac{\cos(\theta - \phi)}{\sin\theta\cos\phi}$.
7. $\cos(A + 45°) + \sin(A - 45°) = 0$.
8. $\sin n\theta\cos\theta + \cos n\theta\sin\theta = \sin(n + 1)\theta$.
9. $\tan\left(\theta - \frac{\pi}{4}\right) + \cot\left(\theta + \frac{\pi}{4}\right) = 0$.
10. From the functions of 30° and 45° calculate the functions of 75°.

For convenience we collect formulas (a), (b) . . . , (h) and form Group B, numbering them consecutively with the formulas of Group A.

Formulas, Group B.

$$\sin(x + y) = \sin x\cos y + \cos x\sin y. \tag{9}$$

$$\cos(x + y) = \cos x\cos y - \sin x\sin y. \tag{10}$$

$$\sin(x - y) = \sin x\cos y - \cos x\sin y. \tag{11}$$

$$\cos(x - y) = \cos x\cos y + \sin x\sin y. \tag{12}$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}. \tag{13}$$

$$\cot(x + y) = \frac{\cot x\cot y - 1}{\cot x + \cot y}. \tag{14}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}. \tag{15}$$

$$\cot(x - y) = \frac{\cot x\cot y + 1}{\cot y - \cot x}. \tag{16}$$

156. Functions of $2x$.—Putting $y = x$ in (9), (10), and (13) of Group B, we have

$$\sin 2x = 2\sin x\cos x, \tag{14}$$

$$\cos 2x = \cos^2 x - \sin^2 x, \tag{15}$$

$$= 1 - 2\sin^2 x,$$

$$= 2\cos^2 x - 1.$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}. \tag{16}$$

For $\cot 2x$ use $\dfrac{1}{\tan 2x}$.

Exercises.

1. Verify these formulas when x is 30°; 45°; 150°; −60°.

Show that,

2. $2 \csc 2x = \sec x \csc x$.

3. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.

4. $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$.

5. $\tan x + \cot x = 2 \csc 2x$.

6. Calculate the functions of $2x$ when $\sin x = \frac{12}{13}$.

Ans. $\sin 2x = \pm \frac{120}{169}$; $\cos 2x = \frac{119}{169}$; etc.

7. Calculate the functions of $2x$ when $\tan x = \frac{3}{4}$.

157. Functions of $\frac{1}{2}x$. — The second and third values of $\cos 2x$ in (15) are

$$\cos 2x = 1 - 2\sin^2 x,$$
$$\cos 2x = 2\cos^2 x - 1.$$

Solving these for $\sin x$ and $\cos x$ respectively, we have

$$\sin x = \pm\sqrt{\frac{1 - \cos 2x}{2}}, \quad \cos x = \pm\sqrt{\frac{1 + \cos 2x}{2}}.$$

Replacing x by $\frac{1}{2}x$, these become

$$(17) \qquad \sin \tfrac{1}{2}x = \pm\sqrt{\frac{1 - \cos x}{2}},$$

$$(18) \qquad \cos \tfrac{1}{2}x = \pm\sqrt{\frac{1 + \cos x}{2}}.$$

Dividing (17) by (18),

$$(19) \qquad \tan \tfrac{1}{2}x = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}.$$

Formulas, Group C.

(14) $\sin 2x = 2 \sin x \cos x$.

(15) $\cos 2x = \cos^2 x - \sin^2 x$
$= 1 - 2\sin^2 x$
$= 2\cos^2 x - 1$.

(16) $\tan 2x = \dfrac{2 \tan x.}{1 - \tan^2 x}$.

(17) $\sin \frac{1}{2}x = \pm\sqrt{\dfrac{1 - \cos x}{2}}$.

(18) $\cos \frac{1}{2}x = \pm\sqrt{\dfrac{1 + \cos x}{2}}$.

(19) $\tan \frac{1}{2}x = \pm\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$
$= \dfrac{1 - \cos x}{\sin x}$
$= \dfrac{\sin x}{1 + \cos x}$.

Exercises.

1. Calculate the functions of $15°$ from those of $30°$.
2. Calculate the functions of $22\frac{1}{2}°$ from those of $45°$.
3. Calculate the functions of $7\frac{1}{2}°$.
4. Calculate the values of $\tan(2x - y)$, when $\sin x = \frac{3}{5}$ and $\cos y = \frac{12}{13}$.

Show that,

5. $\sin 4x = 2 \sin 2x \cos 2x$.

6. $\cos 2x = \dfrac{2 - \sec^2 x}{\sec^2 x}$.

7. $\dfrac{\cos 2x}{1 + \sin 2x} = \dfrac{1 - \tan x}{1 + \tan x}$.

8. $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$.

9. $\dfrac{\cos^3 \theta - \sin^3 \theta}{\cos \theta - \sin \theta} = \dfrac{2 + \sin 2\theta}{2}$.

10. $\cot x + \csc x = \cot \frac{1}{2} x$.

11. $(\sin \frac{1}{2}\theta + \cos \frac{1}{2}\theta)^2 = 1 + \sin \theta$.

12. $\dfrac{1 + \sec \theta}{\sec \theta} = 2 \cos^2 \frac{1}{2}\theta$.

13. $\sin \beta \cot \frac{1}{2}\beta = 1 + \cos \beta$.

14. $1 + \tan \beta \tan \frac{1}{2}\beta = \sec \beta$.

15. $\cot \beta = \dfrac{\cot^2 \frac{\beta}{2} - 1}{2 \cot \frac{\beta}{2}}$.

16. $\dfrac{\cos \beta}{1 - \sin \beta} = \dfrac{1 + \tan \frac{\beta}{2}}{1 - \tan \frac{\beta}{2}}$.

158. Formulas for $\sin u \pm \sin v$ and for $\cos u \pm \cos v$. — Formulas (9) and (11) of Group B are

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y.$$

Adding: $\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$.

Subtracting: $\sin(x + y) - \sin(x - y) = 2 \cos x \sin y$.

Let $x + y = u$, and $x - y = v$;

then $x = \dfrac{u + v}{2}$ and $y = \dfrac{u - v}{2}$.

Substituting in the two preceding equations, we have

$$\sin u + \sin v = 2 \sin \frac{u + v}{2} \cos \frac{u - v}{2}. \tag{20}$$

$$\sin u - \sin v = 2 \cos \frac{u + v}{2} \sin \frac{u - v}{2}. \tag{21}$$

Proceeding similarly with formulas (10) and (12) of Group B, we obtain,

(22) $$\cos u + \cos v = 2 \cos \frac{u+v}{2} \cos \frac{u-v}{2},$$

(23) $$\cos u - \cos v = -2 \sin \frac{u+v}{2} \sin \frac{u-v}{2}.$$

The last four equations, called the addition theorems of trigonometry, we collect as the

Formulas, Group D.

(20) $$\sin u + \sin v = 2 \sin \frac{u+v}{2} \cos \frac{u-v}{2}.$$

(21) $$\sin u - \sin v = 2 \cos \frac{u+v}{2} \sin \frac{u-v}{2}.$$

(22) $$\cos u + \cos v = 2 \cos \frac{u+v}{2} \cos \frac{u-v}{2}.$$

(23) $$\cos u - \cos v = -2 \sin \frac{u+v}{2} \sin \frac{u-v}{2}.$$

Example 1. Show that $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \dfrac{\tan \frac{x+y}{2}}{\tan \frac{x-y}{2}}.$

$$\frac{\sin x + \sin y}{\sin x - \sin y} = \frac{2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}}{2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}}$$

$$= \tan \frac{x+y}{2} \cot \frac{x-y}{2} = \frac{\tan \frac{x+y}{2}}{\tan \frac{x-y}{2}}.$$

Example 2. Show that $\dfrac{\cos 75° + \cos 15°}{\cos 75° - \cos 15°} = -\sqrt{3}.$

$$\frac{\cos 75° + \cos 15°}{\cos 75° - \cos 15°} = \frac{2 \cos 45° \cos 30°}{-2 \sin 45° \sin 30°} = -\cot 45° \cot 30° = -\sqrt{3}.$$

Exercises. Show that:

1. $\sin 3x + \sin 5x = 2 \sin 4x \cos x$.
2. $\sin 10\theta + \sin 6\theta = 2 \sin 8\theta \cos 2\theta$.
3. $\cos 2x + \cos 4x = 2 \cos 3x \cos x$.
4. $\sin 7\alpha - \sin 5\alpha = 2 \cos 6\alpha \sin \alpha$.
5. $\cos 4\theta - \cos 6\theta = 2 \sin 5\theta \sin \theta$.
6. $\cos x + \cos 2x = 2 \cos \frac{3x}{2} \cos \frac{x}{2}$.
7. $\sin 30° + \sin 60° = \sqrt{2} \cos 15°$.
8. $\sin 70° - \sin 10° = \cos 40°$.
9. $\sin 5x \cos 3x = \frac{1}{2}(\sin 8x + \sin 2x)$.
10. $2 \cos 10° \sin 50° = \sin 60° + \sin 40°$.
11. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \dfrac{A+B}{2}$.
12. $\dfrac{\sin \theta + \sin 3\theta}{\cos \theta + \cos 3\theta} = \tan 2\theta$.
13. $2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$.
14. $\sin 4\theta \sin \theta = \frac{1}{2}(\cos 3\theta - \cos 5\theta)$.
15. $\cos 8x - \cos 4x = -4 \sin 2x \sin 3x \cos 3x$.
16. $\sin(2x + 3y) + \sin(2x - 3y) = 2 \sin 2x \cos 3y$.

159. Exercises involving the use of formulas (1) to (23).

1. If $\sin x = \frac{4}{5}$ and $\sin y = \frac{3}{5}$, find the value of $\sin(x+y)$ and $\cos(x+y)$ when x and y are both in the first quadrant.

2. As in exercise 1, when x and y are both in the second quadrant.

3. If $\cos x = \frac{4}{5}$ and $\cos y = \frac{40}{41}$, calculate $\sin(x+y)$ and $\cos(x+y)$ when x and y are both in the first quadrant.

4. As in exercise 3, when x and y are both in the fourth quadrant.

5. If $\sin x = \frac{1}{3}$ and $\sin y = \frac{2}{3}$, calculate all values of $\sin(x \pm y)$.

6. If $\sin \alpha = \frac{1}{4}$ and $\sin \beta = \frac{2}{5}$, calculate all values of $\cos(\alpha \pm \beta)$.

7. If $\cos \alpha = \frac{3}{4}$ and $\cos \beta = \frac{2}{5}$, calculate all values of $\tan(\alpha \pm \beta)$.

8. Calculate $\sin(x+y+z)$ when $\sin x = \frac{5}{13}$, $\sin y = \frac{7}{25}$, $\sin z = \frac{9}{41}$, and x, y, z all lie in the first quadrant.

9. As in exercise 8, when x, y, z all lie in the second quadrant.

10. Calculate $\cos(x+y+z)$ when $\cos x = \frac{4}{5}$, $\cos y = \frac{12}{13}$, $\cos z = \frac{24}{25}$, and x, y, z all lie in the first quadrant.

11. As in exercise 10, when x, y, z all lie in the fourth quadrant.

12. Calculate $\tan(x+y)$ when $\tan x = 1$ and $\cot y = \sqrt{3}$.

13. Calculate all values of $\sin 2(x-y)$ and of $\tan(2x - y)$ when $\tan x = \frac{3}{4}$ and $\tan y = \frac{5}{12}$.

14. Calculate all values of $\cos(\alpha + \beta)$ when $\tan \alpha = m$ and $\tan \beta = n$.

15. Calculate $\cot(\alpha - \beta)$ when $\tan \alpha = a + 1$ and $\tan \beta = a - 1$.

16. Calculate $\tan(\alpha + \beta)$ when $\tan \alpha = \dfrac{x}{x+1}$ and $\tan \beta = \dfrac{1}{2x+1}$.

17. If $\tan \alpha = \frac{1}{7}$ and $\tan \beta = \frac{2}{11}$, calculate $\tan(2\alpha + \beta)$.

18. Calculate sin 75°, cos 75°, and tan 75°, by use of the relation (a) $75° = \frac{150°}{2}$; (b) $75° = 135° - 60°$.

19. Calculate the functions of $202\frac{1}{2}°$; of $7\frac{1}{2}°$.

Prove the following identities:

20. $\sin x \sin (y - z) + \sin y \sin (z - x) + \sin z \sin (x - y) = 0.$

21. $\cos x \sin (y - z) + \cos y \sin (z - x) + \cos z \sin (x - y) = 0.$

22. $\cos (x + y) \cos (x - y) + \sin (y + z) \sin (y - z) - \cos (x + z) \cos (x - z) = 0.$

23. $\cos (x - y + z) = \cos x \cos y \cos z + \cos x \sin y \sin z - \sin x \cos y \sin z + \sin x \sin y \cos z.$

24. $\sin 3x = 3 \sin x - 4 \sin^3 x.$

25. $\cos 3x = 4 \cos^3 x - 3 \cos x.$

26. $\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}.$

27. $\cot 3x = \frac{\cot^3 x - 3 \cot x}{3 \cot^2 x - 1}.$

28. $\tan 4\theta = \frac{4 \tan \theta (1 - \tan^2 \theta)}{1 - 6 \tan^2 \theta + \tan^4 \theta}.$

29. $\frac{\sin (\alpha + \beta)}{\cos \alpha \cos \beta} = \tan \alpha + \tan \beta.$

30. $\frac{\cos (\alpha + \beta)}{\sin \alpha \cos \beta} = \cot \alpha - \tan \beta.$

31. $\frac{\cos (\alpha - \beta)}{\cos \alpha \sin \beta} = \cot \beta + \tan \alpha.$

32. $\frac{\sin (\alpha - \beta)}{\cos \alpha \cos \beta} = \tan \alpha - \tan \beta.$

33. $\frac{\sin (x + y)}{\sin (x - y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$

34. $\frac{\cos (x + y)}{\cos (x - y)} = \frac{\cot x - \tan y}{\cot x + \tan y}.$

35. $\sin (\theta + \phi) \sin (\theta - \phi) = \cos^2 \phi - \cos^2 \theta.$

36. $\cos (u + v) \cos (u - v) = \cos^2 u - \sin^2 v.$

37. $\sin (A - 45°) = \frac{1}{\sqrt{2}} (\sin A - \cos A).$

38. $\cot \left(A - \frac{\pi}{4}\right) = \frac{\cot A + 1}{1 - \cot A}.$

39. $\tan \left(\theta - \frac{\pi}{4}\right) = \frac{\tan \theta - 1}{\tan \theta + 1}.$

40. $\tan \left(\frac{\pi}{4} + \theta\right) = \frac{1 + \tan \theta}{1 - \tan \theta}.$

41. $\tan \left(\alpha + \frac{\pi}{3}\right) + \tan \left(\alpha - \frac{\pi}{3}\right) = \frac{8 \cot \alpha}{\cot^2 \alpha - 3}.$

42. $\frac{\sin \frac{5\pi}{12}}{\sin \frac{\pi}{12}} - \frac{\cos \frac{5\pi}{12}}{\cos \frac{\pi}{12}} = 2\sqrt{3}.$

43. $\tan \left(\frac{\pi}{4} + \theta\right) = \frac{1}{\tan \left(\frac{\pi}{4} - \theta\right)}.$

44. $\cos \left(\theta + \frac{\pi}{4}\right) + \sin \left(\theta - \frac{\pi}{4}\right) = 0.$

45. $\cot \left(\theta + \frac{\pi}{4}\right) + \tan \left(\theta - \frac{\pi}{4}\right) = 0.$

46. $\cot \left(\theta - \frac{\pi}{4}\right) + \tan \left(\theta + \frac{\pi}{4}\right) = 0.$

47. $\cot \frac{\pi}{8} - \tan \frac{\pi}{8} = 2.$

48. $2 \cos \frac{\pi}{8} = \sqrt{2 + \sqrt{2}}.$

49. $\sqrt{2} \sin (\theta + 45°) = \sin \theta + \cos \theta.$

50. $\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}.$

51. $\sec 2x = \frac{\csc^2 x}{\csc^2 x - 2}.$

52. $\cot \theta - \cot 2\theta = \csc 2\theta.$

53. $\sec^2 \theta \cos 2\theta = 1 - \tan^2 \theta.$

54. $1 + \tan \theta \tan 2\theta = \sec 2\theta.$

55. $1 - \cos 2x = \tan x \sin 2x.$

56. $\sec 2\theta = \frac{\cot^2 \theta + 1}{\cot^2 \theta - 1}.$

57. $\frac{\sin 2\theta}{1 + \cos 2\theta} = \tan \theta.$

58. $\frac{\sin 2\theta}{1 - \cos 2\theta} = \cot \theta.$

59. $\cot^2\theta - 1 = 2\cot\theta\cot 2\theta.$

60. $2 - \sec^2\theta = \sec^2\theta\cos 2\theta.$

61. $\dfrac{\cos 2\theta}{1+\sin 2\theta} = \dfrac{1-\tan\theta}{1+\tan\theta}.$

62. $\dfrac{\cos 3x}{\cos x} = 2\cos 2x - 1.$

63. $\tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x}.$

64. $\dfrac{\cos 3\theta}{\sin\theta} + \dfrac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta.$

65. $\dfrac{\tan\theta + \cot\theta}{\cot\theta - \tan\theta} = \sec 2\theta.$

66. $\tan(45° + \phi) - \tan(45° - \phi) = 2\tan 2\phi.$

67. $\dfrac{\cos^3\phi + \sin^3\phi}{\cos\phi + \sin\phi} = \dfrac{2 - \sin 2\phi}{2}.$

68. $\dfrac{\cos^5\phi - \sin^5\phi}{\cos\phi - \sin\phi} = 1 + \tfrac{1}{2}\sin 2x - \tfrac{1}{4}\sin^2 2x.$

69. $\dfrac{\sin x + \cos x}{\cos x - \sin x} = \tan 2x - \sec 2x.$

70. $\sin 2x \tan 2x = \dfrac{4\tan^2 x}{1 - \tan^4 x}.$

71. $\cos^2\theta + \sin^2\theta\cos 2\phi = \cos^2\phi + \sin^2\phi\cos 2\theta.$

72. $1 + \cos 2(\theta - \phi)\cos 2\phi = \cos^2\theta + \cos^2(\theta - 2\phi).$

73. $\dfrac{\tan^2\left(\theta + \frac{\pi}{4}\right) - 1}{\tan^2\left(\theta + \frac{\pi}{4}\right) + 1} = \sin 2\theta.$

74. $\dfrac{\cos\left(x + \frac{\pi}{4}\right)}{\cos\left(x - \frac{\pi}{4}\right)} = \sec 2x - \tan 2x.$

75. $\tan x = \dfrac{\sin x + \sin 2x}{1 + \cos x + \cos 2x}.$

76. $\tan x = \dfrac{\sin 2x - \sin x}{1 - \cos x + \cos 2x}.$

77. $\sec 2\theta - \tfrac{1}{2}\tan 2\theta\sin 2\theta = \dfrac{\cot^2\theta + \tan^2\theta}{\cot^2\theta - \tan^2\theta}.$

78. $\dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta} = \sqrt{\dfrac{1+\sin 2\theta}{1 - \sin 2\theta}}.$

79. $\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2 = 1 + \sin\theta.$

80. $\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2 = 1 - \sin\theta.$

81. $\dfrac{\cos\theta}{1 - \sin\theta} = \dfrac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}.$

82. $\dfrac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}} = \sec x - \tan x.$

83. $\tan x - \tan\frac{x}{2} = \tan\frac{x}{2}\sec x.$

84. $\dfrac{1 + \sec\phi}{\sec\phi} = 2\cos^2\dfrac{\phi}{2}.$

85. $\sec^2\dfrac{x}{2} = 2\tan\dfrac{x}{2}\csc x.$

86. $\dfrac{1 + \cos 3\phi}{\sin 3\phi} = \cot\dfrac{3\phi}{2}.$

87. $\dfrac{1 + \sin 45°}{\cos 45°} = \tan 67\tfrac{1}{2}°.$

88. $\dfrac{1}{\sec\theta + \tan\theta} = \cot\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right).$

89. $\dfrac{1 + \sin x + \cos x}{1 + \sin x - \cos x} = \cot\dfrac{x}{2}.$

90. $\tan\dfrac{x}{2} = \sqrt{\dfrac{2\sin x - \sin 2x}{2\sin x + \sin 2x}}.$

91. $\sqrt{3}\sin 75° - \cos 75° = \sqrt{2}.$

92. $\sin\frac{5\theta}{2}\cos\frac{\theta}{2} - \sin\frac{9\theta}{2}\cos\frac{3\theta}{2} + \cos 4\theta\sin 2\theta = 0.$

93. $\sin 4x + \sin 2x = 2\sin 3x\cos x.$

94. $\sin 3x + \sin 5x = 8\sin x\cos^2 x\cos 2x.$

95. $\dfrac{\cot 15^\circ + \tan 15^\circ}{\cot 15^\circ - \tan 15^\circ} = \dfrac{2}{\sqrt{3}}.$

96. $\dfrac{1 - \sqrt{2}\sin 75^\circ}{1 - \sqrt{2}\cos 75^\circ} = -\cot 60^\circ.$

97. $\sin 100^\circ - \sin 40^\circ = \sin 20^\circ.$

98. $\cos\left(\dfrac{\pi}{3} + \alpha\right) + \cos\left(\dfrac{\pi}{3} - \alpha\right) = \cos\alpha.$

99. $\cos\left(\dfrac{\pi}{4} + \alpha\right) + \cos\left(\dfrac{\pi}{4} - \alpha\right) = \sqrt{2}\cos\alpha.$

100. $\cos(\theta + \phi) - \sin(\theta - \phi) = 2\sin\left(\dfrac{\pi}{4} - \theta\right)\cos\left(\dfrac{\pi}{4} - \phi\right).$

101. $2\sin\left(\alpha + \dfrac{\pi}{4}\right)\sin\left(\alpha - \dfrac{\pi}{4}\right) = \sin^2\alpha - \cos^2\alpha.$

102. $\sin\left(\dfrac{\pi}{4} + \alpha\right) - \sin\left(\dfrac{\pi}{4} - \alpha\right) = \sqrt{2}\sin\alpha.$

103. $\sin 40^\circ - \sin 10^\circ = \dfrac{\sqrt{3} - 1}{\sqrt{2}}\cos 25^\circ.$

104. $\sin 3x + \sin x = 4\sin x\cos^2 x.$

105. $\dfrac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \sqrt{3}.$

106. $\dfrac{\cos x + \cos y}{\cos x - \cos y} = -\cot\dfrac{x+y}{2}\cot\dfrac{x-y}{2}.$

107. $\dfrac{\sin 70^\circ + \sin 20^\circ}{\cos 70^\circ + \cos 20^\circ} = 1.$

108. $\dfrac{\sin 100^\circ + \sin 40^\circ}{\sin 100^\circ - \sin 40^\circ} = \sqrt{3}\tan 70^\circ.$

109. $\dfrac{(\sin\alpha + \sin\beta)(\cos\alpha + \cos\beta)}{(\sin\alpha - \sin\beta)(\cos\alpha - \cos\beta)} = -\cot^2\dfrac{\alpha - \beta}{2}.$

110. $\dfrac{(\sin\alpha + \sin\beta)(\cos\alpha - \cos\beta)}{(\sin\alpha - \sin\beta)(\cos\alpha + \cos\beta)} = -\tan^2\dfrac{\alpha + \beta}{2}.$

111. $\dfrac{(\sin 75^\circ + \sin 15^\circ)(\cos 75^\circ + \cos 15^\circ)}{(\sin 75^\circ - \sin 15^\circ)(\cos 75^\circ - \cos 15^\circ)} = -3.$

112. $\dfrac{\cos 2x + \cos 12x}{\cos 6x + \cos 8x} + \dfrac{\cos 7x - \cos 3x}{\cos x - \cos 3x} + \dfrac{2\sin 4x}{\sin 2x} = 0.$

113. $\sin x + \sin 2x + \sin 3x = 4\cos\frac{1}{2}x\cos x\sin\frac{3}{2}x.$

(*Hint.* Replace $\sin x + \sin 3x$ by $2\sin 2x\cos x$ and $\sin 2x$ by $2\sin x\cos x$; from these results factor out $2\cos x$ and combine the remainders by the formula for $\sin u + \sin v$.)

114. $\cos x + \cos 2x + \cos 3x = 4\cos\frac{1}{2}x\cos x\cos\frac{3}{2}x - 1.$

115. $\sin 2x + \sin 4x + \sin 6x = 4\cos x\cos 2x\sin 3x.$

116. $\dfrac{\sin\theta + \sin 2\theta + \sin 3\theta}{\cos\theta + \cos 2\theta + \cos 3\theta} = \tan 2\theta.$

117. $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0.$

118. $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 4\cos\theta\cos 2\theta\cos 4\theta.$

119. $\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 16\sin\theta\cos^2\theta\cos^2 2\theta.$

120. $4\sin^2\phi\cos^2\phi + (\cos^2\phi - \sin^2\phi)^2 = 1.$

121. $(\cos x\cos y + \sin x\sin y)^2 + (\sin x\cos y - \cos x\sin y)^2 = 1.$

122. $\dfrac{\tan 3x - \tan x}{1 + \tan 3x\tan x} = \tan 2x.$

123. $\dfrac{\tan(n+1)\theta - \tan n\theta}{1 + \tan(n+1)\theta\tan n\theta} = \tan\theta.$

124. $\dfrac{\tan(\theta+\phi)-\tan\phi}{1+\tan(\theta+\phi)\tan\phi}=\tan\theta.$

125. $\dfrac{\tan(\theta-\phi)+\tan\phi}{1-\tan(\theta-\phi)\tan\phi}=\tan\theta.$

126. $\sin n\theta\cos\theta+\cos n\theta\sin\theta=\sin(n+1)\theta.$

127. $2\csc 4x-2\cot 4x=\cot x-\tan x.$

128. $\dfrac{1-\cos 3x}{1-\cos x}=(1+\cos 2x)^2.$ **129.** $\dfrac{\sin 3\theta}{\sin\theta}-\dfrac{\cos 3\theta}{\cos\theta}=2.$

130. If $\tan x=\dfrac{b}{a}$, show that $\sqrt{\dfrac{a+b}{a-b}}+\sqrt{\dfrac{a-b}{a+b}}=\dfrac{2\cos x}{\sqrt{\cos 2x}}.$

131. $4\cos^3 x\sin 3x+4\sin^3 x\cos 3x=3\sin 4x.$

132. $\sin^3 x+\sin^3(120°+x)+\sin^3(240°+x)=-\frac{3}{4}\sin 3x.$

133. $\cos 6x=16(\cos^6 x-\sin^6 x)-15\cos 2x.$

134. $1+\tan^6 x=\sec^4 x(\sec^2 x-3\sin^2 x).$

135. $\dfrac{3\sin x-\sin 3x}{3\cos x+\cos 3x}=\tan^3 x.$

136. $\sin 2x\sin 2y=\sin^2(x+y)-\sin^2(x-y).$

137. $\sin 5\alpha\sin\alpha=\sin^2 3\alpha-\sin^2 2\alpha.$

138. $\cos^4\alpha=\frac{1}{8}(3+4\cos 2\alpha+\cos 4\alpha).$

139. $\cos 2x+\cos 2y+\cos 2z+\cos 2(x+y+z)=4\cos(x+y)\cos(y+z)\cos(z+x).$

140. $\sin^2 x+\sin^2 y+\sin^2 z+\sin^2(x+y+z)=2-2\cos(x+y)\cos(y+z)\cos(z+x).$

141. $\cos^2 x+\cos^2 y+\cos^2 z+\cos^2(x+y-z)=2+2\cos(x+y)\cos(x-z)\cos(y-z).$

142. $\sin(x-y-z)-\sin x-\sin y-\sin z=4\sin\dfrac{x-y}{2}\sin\dfrac{x-z}{2}\sin\dfrac{y+z}{2}.$

143. $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=\sin 2(\alpha+\beta+\gamma)+4\sin(\alpha+\beta)\sin(\beta+\gamma)\sin(\alpha+\gamma).$

144. $\sin(\alpha+\beta-\gamma)+\sin(\alpha-\beta+\gamma)+\sin(\beta+\gamma-\alpha)-\sin(\alpha+\beta+\gamma)=4\sin\alpha\sin\beta\sin\gamma.$

145. $\cos(\alpha+\beta-\gamma)+\cos(\beta+\gamma-\alpha)+\cos(\alpha+\gamma-\beta)-\cos(\alpha+\beta+\gamma)=4\cos\alpha\cos\beta\cos\gamma.$

146. Show that the equation $\sin x=a+\dfrac{1}{a}$ is impossible.

147. For what values of a will the equation $2\cos x=a+\dfrac{1}{a}$ give possible values for x?

148. Show that $2\sin\dfrac{x}{2}=-\sqrt{1+\sin x}-\sqrt{1-\sin x}$, provided that x lies in the second or third quadrant.

149. Show that $2\cos\dfrac{x}{2}=-\sqrt{1+\sin x}+\sqrt{1-\sin x}$, provided that x lies in the second or third quadrant.

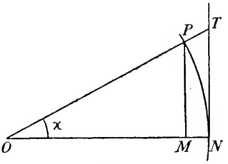
150. When x lies in the fourth quadrant, show that

$$2\sin\frac{x}{2}=\sqrt{1-\sin x}-\sqrt{1+\sin x}.$$

CHAPTER IX

RATIOS $\frac{\sin x}{x}$ AND $\frac{\tan x}{x}$. INVERSE FUNCTIONS. TRIGONOMETRIC EQUATIONS

160. The limits of the ratios $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$. Let $x = \angle NOP$ (figure) lie between 0° and 90°; let NP be a circular arc with center at O, and MP and $NT \perp ON$. Then



$$MP < NP < NT;$$

hence $$\frac{MP}{OP} < \frac{NP}{OP} < \frac{NT}{OP},$$

or $$\sin x < x \text{ (radians)} < \tan x.$$

That is, *the radian measure of any acute angle lies between the sine and the tangent of the angle.*

From the last inequality we have, on dividing by $\sin x$,

$$1 < \frac{x}{\sin x} < \sec x.$$

Suppose x to decrease and approach 0. Then $\sec x \doteq 1$, and consequently also $\frac{x}{\sin x} \doteq 1$ and $\frac{\sin x}{x} \doteq 1$.

Hence $$\lim_{x \doteq 0} \frac{\sin x}{x} = 1.$$

Dividing the third of the above inequalities by $\tan x$, we have

$$\cos x < \frac{x}{\tan x} < 1;$$

letting x approach zero we have

$$\lim_{x = 0} \frac{\tan x}{x} = 1$$

Hence, *the ratio of either the sine or the tangent to the angle* (*in radians*) *approaches* 1 *as its limit when the angle approaches zero.*

When angle x is small, these ratios will be nearly equal to 1; that is,

$$\frac{\sin x}{x} = 1 + e \quad \text{and} \quad \frac{\tan x}{x} = 1 + e_1,$$

where e and e_1 are small quantities. Hence

$$\sin x = x + ex \quad \text{and} \quad \tan x = x + e_1 x.$$

Neglecting the small terms ex and $e_1 x$, we have

$$\sin x = \tan x = x \textit{ approximately}, \text{ when } x \text{ is small.}$$

Hence *when x is small, $\sin x$ and $\tan x$ are nearly equal to x (in radians).*

The degree of this approximation is indicated by the following values:

Angle x. Degrees	radians	$\sin x$	$\tan x$
1°	.0174532925+	.0174524064+	.0174550649+
1′	.0002908882+	.0002908882+	.0002908882+
1″	.0000048481+	.0000048481+	.0000048481+

Exercises.

1. How large may x be if the approximations

$$\sin x = x \quad \text{and} \quad \tan x = x$$

are to be correct to four places inclusive? (Table.)

2. In what decimal place is the error of the *approximations*

$$\sin 30° = 30 \sin 1° \quad \text{and} \quad \tan 30° = 30 \tan 1°?$$

3. How large may n be if the approximations

$$\sin n° = n \sin 1° \quad \text{and} \quad \tan n° = n \tan 1°$$

are to be correct to three decimals inclusive?

4. As in exercise 3, for the approximations

$$\sin n' = n \sin 1' \quad \text{and} \quad \tan n' = n \tan 1'.$$

161. Inverse Trigonometric Functions. — It is often convenient to specify an angle, not by its degree or radian measure, but by the value of one of its functions. Thus we may speak of 30° as "an angle whose sine is $\frac{1}{2}$." There is of course an ambiguity here, since 30° is only one of the angles whose sine is $\frac{1}{2}$.

If x is an angle whose sine is a, we write

$$x = \sin^{-1} a,$$

which may be read "x equals an angle whose sine is a," or "x equals the inverse sine of a," or "x equals anti-sine a."

Similarly the equation

$$x = \tan^{-1} a$$

is read "x equals an angle whose tangent is a," or "x equals the inverse tangent of a," or "x equals anti-tangent a," and so on for the other functions.

Obviously the equations

$$x = \sin^{-1} a \quad \text{and} \quad \sin x = a$$

are equivalent. Similarly for

$$x = \tan^{-1} a \quad \text{and} \quad \tan x = a,$$

$$x = \sec^{-1} a \quad \text{and} \quad \sec x = a,$$

and so on.

It should be noted that "-1" in $\sin^{-1} a$ is *not an exponent;* it might equally well have been written as a subscript, $\sin_{-1} x$, or in any other convenient way. The reason for writing it as above will appear by noting that, according to the laws of exponents, the algebraic equations

$$x = b^{-1} a \quad \text{and} \quad bx = a$$

are equivalent.

When it is necessary to write $\sin x$ with an exponent -1, it should be written $(\sin x)^{-1}$, *not* $\sin^{-1} x$.

The smallest positive angle whose sine is a is often called the *principal value* of the symbol $\sin^{-1} a$. Similarly for the other functions.

If θ denote the principal value of any inverse function, we have from (**146**), equations I, II, III,

$$\sin^{-1} a = 2n\pi + \theta, \text{ or } (2n+1)\pi - \theta; \qquad \csc^{-1} a = 2n\pi + \theta, \text{ or } (2n+1)\pi - \theta;$$

$$\cos^{-1} a = 2n\pi \pm \theta; \qquad \sec^{-1} a = 2n\pi \pm \theta;$$

$$\tan^{-1} a = n\pi + \theta; \qquad \cot^{-1} a = n\pi + \theta.$$

162. Equations Involving Inverse Functions. — In this article we shall restrict the symbol for the inverse functions to mean *only the principal value* of the function. Thus, $\sin^{-1}\frac{1}{2}$ shall mean the angle 30° only, $\tan^{-1}1 = 45°$, and so on.

Example 1. Show that $\sin^{-1}\frac{3}{5} = \cos^{-1}\frac{4}{5}$.

Let $x = \sin^{-1}\frac{3}{5}$ and $y = \cos^{-1}\frac{4}{5}$;

to prove that $x = y$,

or that $\sin x = \sin y$.

(We use the sine for convenience; any other function might be used.)

Now $\sin x = \frac{3}{5}$ since $x = \sin^{-1}\frac{3}{5}$.

Also $\cos y = \frac{4}{5}$; hence $\sin y = \sqrt{1 - \cos y^2} = \frac{3}{5}$. q. e. d.

Example 2. Show that $2\tan^{-1}2 = \sin^{-1}\frac{4}{5}$.

Let $x = \tan^{-1}2$ and $y = \sin^{-1}\frac{4}{5}$;

to prove that $2x = y$,

or that $\sin 2x = \sin y$.

Now $\sin 2x = 2\sin x\cos x$.

But $\tan x = 2$; hence $\sin x = \frac{2}{\sqrt{5}}$ and $\cos x = \frac{1}{\sqrt{5}}$. **(149.)**

Therefore $\sin 2x = \frac{4}{5} = \sin y$. q. e. d.

Observe that if x were not restricted to be the principal value of $\tan^{-1}2$, we might have $\sin x = -\frac{2}{\sqrt{5}}$.

Example 3. Show that $\tan^{-1}\frac{2}{3} + \tan^{-1}2 + \tan^{-1}8 = \pi$.

Let $x = \tan^{-1}\frac{2}{3}$; $y = \tan^{-1}2$; $z = \tan^{-1}8$;

then $\tan x = \frac{2}{3}$; $\tan y = 2$; $\tan z = 8$.

To prove that $x + y + z = \pi$,

or that $x + y = \pi - z$,

or that $\tan(x + y) = \tan(\pi - z) = -\tan z$.

Now $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x\tan y} = \frac{\frac{2}{3} + 2}{1 - \frac{4}{3}} = -8 = -\tan z$. q. e. d.

Example 4. Show that $\tan^{-1}a = \sin^{-1}\frac{a}{\sqrt{1 + a^2}}$ when $a > 0$.

Let $x = \tan^{-1}a$ and $y = \sin^{-1}\frac{a}{\sqrt{1 + a^2}}$;

then $\tan x = a$ and $\sin y = \frac{a}{\sqrt{1 + a^2}}$.

To prove that $x = y,$

or that $\sin x = \sin y.$

Now since x and y stand for principal values, and a is positive, both angles are in the first quadrant.

Then from $\tan x = a$ we find (**149**)

$$\sin x = \frac{a}{\sqrt{1 + a^2}},$$

which is $\sin y$. q. e. d.

Discuss the above example when the symbol for the inverse functions is assumed to stand for all angles having the function in question, instead of the principal value only.

163. Exercises.

1. Show that the equation in example 4 is not true for principal values when a is negative. (Try $a = -1$.)

Prove the following:

2. $\tan^{-1}\frac{5}{7} + \tan^{-1}\frac{1}{6} = \frac{\pi}{4}.$

3. $2\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{4}{3}.$

4. $\tan^{-1}3 + \frac{\pi}{4} = \tan^{-1}(-2).$

5. $\tan^{-1}\frac{1}{2} + \csc^{-1}\sqrt{10} = \frac{\pi}{4}.$

6. $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2} = 120°.$

7. $2\tan^{-1}3 = \sin^{-1}\frac{3}{5}.$

8. $3\sin^{-1}\frac{1}{3} = \sin^{-1}\frac{23}{27}.$

9. $2\cot^{-1}2 = \csc^{-1}\frac{5}{4}.$

10. $4\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{1}{239} + \frac{\pi}{4}.$

11. $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} + \tan^{-1}\left(\frac{-1}{2}\right) = \pi.$

12. $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{13}{85} = \frac{\pi}{2}.$

13. $\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}.$

14. $2\tan^{-1}\frac{2}{3} - \csc^{-1}\frac{5}{3} = \sin^{-1}\frac{33}{65}.$

15. $\sin^{-1}a = \cos^{-1}\sqrt{1 - a^2}$, if $a > 0$.

16. $2\tan^{-1}m = \tan^{-1}\frac{2m}{1 - m^2}.$

17. $2\tan^{-1}(\cos 2\theta) = \tan^{-1}\left(\frac{\cot^2\theta - \tan^2\theta}{2}\right).$

164. Trigonometric Equations. — A trigonometric equation is an equation which involves one or more trigonometric functions of one or more angles. Thus:

$$\sin^2 x + \cos x = 1;\ \tan\theta + \sec\theta = 3;\ \cot\alpha\csc\alpha = 2.$$

To find the values of the angle which satisfy such an equation, it is usually best to use a method adapted to the case in hand. We give here one general rule, which covers a considerable variety of cases.

Rule: To solve a trigonometric equation, express all its terms by means of a single function; solve as an algebraic equation, considering this function as unknown; find the angles corresponding to the values of the function so obtained. *Check all answers by substitution.*

Examples.

1. $\sin^2 x + \cos x = 1.$

Expressing all terms by means of $\cos x$, we have

$$1 - \cos^2 x + \cos x = 1, \quad \text{or} \quad \cos^2 x - \cos x = 0.$$

$$\therefore \quad \cos x = 0, \quad \text{or} \quad \cos x = 1.$$

Hence x may be any odd multiple of $\frac{\pi}{2}$ or any multiple of 2π; i.e., if n be any integer or zero,

$$x = \pm (2n + 1)\frac{\pi}{2} \quad \text{or} \quad x = \pm 2n\pi.$$

Exercise. Check these answers by substitution.

2. $\tan\theta + \sec\theta = 3.$

Expressing all terms by means of $\tan\theta$, we have

$$\tan\theta \pm \sqrt{1 + \tan^2\theta} = 3, \text{ or } \pm\sqrt{1 + \tan^2\theta} = 3 - \tan\theta.$$

Squaring and reducing,

$$\tan\theta = \frac{4}{3}; \text{ hence } \theta = 53° \, 8' \pm n\pi.$$

When n is odd, these values of θ do not satisfy the given equation. Hence the solutions are

$$\theta = 53° \, 8' \pm 2n\pi.$$

3. $\cot\alpha \csc\alpha = 2.$

Then $\quad \pm \cot\alpha\sqrt{1 + \cot^2\alpha} = 2, \quad \text{or} \quad \cot^4\alpha + \cot^2\alpha = 4.$

Hence $\quad \cot\alpha = \pm\sqrt{-\frac{1}{2} \pm \frac{1}{2}\sqrt{17}}.$

Using the upper sign under the radical (the lower sign makes α imaginary), we have

$$\cot\alpha = \pm 1.2496+; \quad \text{hence} \quad \alpha = \pm 38° \, 40' \pm n\pi.$$

When n is odd, the values of α must be discarded. Hence

$$\alpha = \pm 38° \, 40' \pm 2n\pi.$$

The reason for the additional values in the last two examples is that in example 2 we really solved both the equations $\tan\theta \pm \sec\theta = 3$, and in example 3, both the equations $\cot\alpha \csc\alpha = \pm 2$.

165. *Examples Illustrating Special Methods.* — These depend chiefly on transforming the given equation by means of some of the standard formulas.

4. $2\sin^2 x - 3\sin x\cos x = 1.$

Since $2\sin^2 x = 1 - \cos 2x$ and $2\sin x\cos x = \sin 2x$, we have

$$1 - \cos 2x - \frac{3}{2}\sin 2x = 1, \quad \text{or} \quad \tan 2x = -\frac{2}{3}.$$

Hence $$2x = \tan^{-1}\left(-\frac{2}{3}\right) = -33^\circ\ 41' \pm n\pi.$$

$$\therefore \quad x = -16^\circ\ 50'.5 \pm n\frac{\pi}{2}.$$

Exercise. Check these answers. Solve the given equation by expressing $\cos x$ in terms of $\sin x$.

5. $\sin 3y - \sin 2y = 0.$

By formula (21) of (**158**) this becomes

$$2\cos\frac{5}{2}y\sin\frac{1}{2}y = 0.$$

Hence $$\cos\frac{5}{2}y = 0 \quad \text{or} \quad \sin\frac{1}{2}y = 0.$$

$$\therefore \quad \frac{5}{2}y = \pm(2n+1)\frac{\pi}{2}, \quad \text{or} \quad \frac{1}{2}y = \pm n\pi.$$

$$\therefore \quad y = \pm(2n+1)\frac{\pi}{5}, \quad \text{or} \quad y = \pm 2n\pi.$$

6. $\cos x + \cos 3x + \cos 5x = 0.$

Since $\cos x + \cos 5x = 2\cos 3x\cos 2x$, we have

$$2\cos 3x\cos 2x + \cos 3x = 0, \quad \text{or} \quad \cos 3x(2\cos 2x + 1) = 0.$$

Hence $$\cos 3x = 0, \quad \text{or} \quad \cos 2x = -\frac{1}{2}.$$

$$3x = \pm(2n+1)\frac{\pi}{2}, \quad \text{or} \quad 2x = \pm\frac{2\pi}{3} \pm 2n\pi.$$

$$\therefore \quad x = \pm(2n+1)\frac{\pi}{6} \quad \text{or} \quad \pm\frac{\pi}{3} \pm n\pi.$$

7. $\tan 4\alpha\tan 5\alpha = 1.$

This may be written $\tan 4\alpha = \cot 5\alpha$. But when the tangent of an angle A equals the cotangent of an angle B, $A + B$ must be an odd multiple of $\frac{\pi}{2}$.

Hence $$4\alpha + 5\alpha = \pm(2n+1)\frac{\pi}{2}$$

$$\therefore \quad \alpha = \pm(2n+1)\frac{\pi}{18}.$$

Here α is any odd multiple of 10°.

Otherwise thus: $\tan 4\alpha - \cot 5\alpha = 0$; hence $\frac{\sin 4\alpha}{\cos 4\alpha} - \frac{\cos 5\alpha}{\sin 5\alpha} = 0$;

or $$\frac{\sin 4\alpha\sin 5\alpha - \cos 4\alpha\cos 5\alpha}{\cos 4\alpha\sin 5\alpha} = -\frac{\cos 9\alpha}{\cos 4\alpha\sin 5\alpha} = 0.$$

$$\therefore \quad \cos 9\alpha = 0, \quad \text{or} \quad 9\alpha = \pm(2n+1)\frac{\pi}{2}.$$

Exercise 1. Check these answers. Draw figures for several values of α as 10°, 30°, 50°, 70°. Discuss the case $\alpha = 90°$.

Exercise 2. In example 7, in passing from the first equation to the second we divide by $\tan 5\alpha$, which is permissible only if $\tan 5\alpha \neq 0$. Justify the division.

Exercise 3. Justify the division by $\cos x$ in example 4.

8. $a \sin\theta + b\cos\theta = c.$

We might reduce to $\sin\theta$ or $\cos\theta$ and proceed according to the rule of (**164**). A method much preferred in practice is as follows.

In place of a and b introduce two new constants m and M such that

$$\begin{cases} a = m\cos M, \\ b = m\sin M; \end{cases} \quad \text{whence} \quad \begin{cases} m = \sqrt{a^2+b^2} \\ M = \tan^{-1}\dfrac{b}{a}. \end{cases}$$

The given equation then becomes

$$m(\sin\theta\cos M + \cos\theta\sin M) = C \quad \text{or} \quad \sin(\theta + M) = \frac{c}{m}.$$

Hence if we let $\sin^{-1}x$ represent all angles whose sine is x,

$$\theta + M = \sin^{-1}\frac{c}{m}, \quad \text{or} \quad \theta = \sin^{-1}\frac{c}{m} - M.$$

$$\therefore \quad \theta = \sin^{-1}\frac{c}{\sqrt{a^2+b^2}} - \tan^{-1}\frac{b}{a}.$$

Graphic Solution. As an example, we take the equation

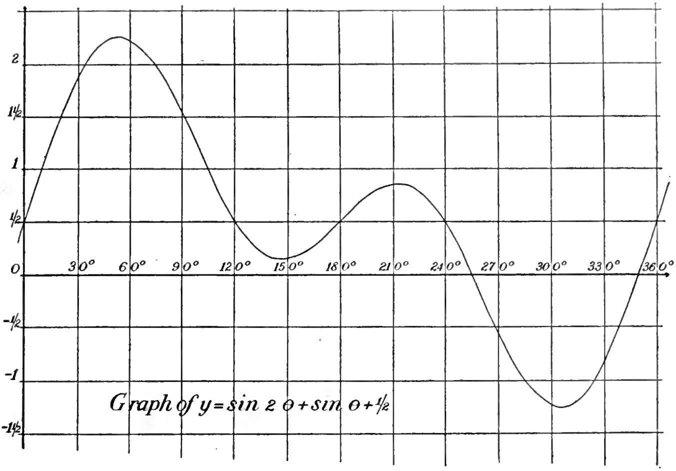
$$\sin 2\theta + \sin\theta + \frac{1}{2} = 0.$$

We want the values of θ which reduce the expression $\sin 2\theta + \sin\theta + \frac{1}{2}$ to zero.

Let $$y = \sin 2\theta + \sin\theta + \frac{1}{2}.$$

Calculate y for a series of values of θ, as $\theta = 0°, 10°, 20°, \ldots,$ and plot the points (θ, y) in rectangular coördinates. The resulting curve will show the approximate values of θ for which y is zero. Any convenient scales may be used on the axes of θ and y.

Let the student read off the required solutions from the graph below.



Exercise. By means of this graph solve the equations

(a) $\sin 2\theta + \sin\theta = 0$;

(b) $\sin 2\theta + \sin\theta = 1$;

(c) $\sin 2\theta + \sin\theta = \frac{1}{4}$.

166. Exercises. Solve the following equations:

1. $2\sin^2 x - 3\cos x = 0$.
2. $4\sin^2\alpha + 1 = 8\cos\alpha$.
3. $\sin\alpha + \cos\alpha = \sqrt{2}$.
4. $\tan\theta + \cot\theta = 2$.
5. $\tan\beta + 3\cot\beta = 4$.
6. $2\sin^2 x + 3 = 5\sin x$.
7. $2(1 - \sin\theta) = \cos\theta$.
8. $5\sin\theta + 10\cos\theta = 11$.
9. $\cos 2x = \cos^2 x$.
10. $2\cos 2x = 1 + 2\sin x$.
11. $4\cot 2\theta = \cot^2\theta - \tan^2\theta$.
12. $\cos\theta = \sin 2\theta$.
13. $\tan 2x = 3\tan x$.
14. $\sin 2y = \cos 3y$.
15. $\tan\alpha = \cot 3\alpha$.
16. $\cot 8\phi = \tan\phi$.
17. $\sec px = \csc qx$.
18. $\tan y = \cot 6y$.
19. $\sin r\theta = \sin s\theta$.
20. $\cot(30° - x) = \tan(30° + 3x)$.
21. $1 + \tan\beta = \tan\left(\frac{\pi}{4} + \beta\right)$.
22. $\sin 4\alpha = \cos 5\alpha$.
23. $\sin(60° - x) - \sin(60° + x) = \frac{1}{2}\sqrt{3}$.
24. $\sin 2\theta + \sin 4\theta = \sqrt{2}\cos\theta$.
25. $\sin(30° + \theta) - \cos(60° + \theta) = -\frac{1}{2}\sqrt{3}$.
26. $\sin 4\alpha = \cos 3\alpha + \sin 2\alpha$.
27. $\sin 3\beta + \sin\beta = \cos\beta - \cos 3\beta$.
28. $\sin x + \sin 2x + \sin 3x = 0$.
29. $\sin x + \sin 3x + \sin 5x = 0$.
30. $\cos x + \cos 2x = \cos\frac{1}{2}x$.

Solve some of the above equations graphically, in particular 1, 2, 4, 5, 7, 8, 12, 13, 14, 15, 26, 28, 29.

167. Simultaneous Trigonometric Equations. — We shall now give some examples to illustrate methods for solving a system of simultaneous trigonometric equations for several unknown quantities. To express answers concisely, we shall now use the symbols for the inverse functions to mean *all* the angles determined by the given function.

Examples.

1. Solve for r and θ:

$$r\cos\theta = x,$$
$$r\sin\theta = y.$$

Squaring and adding, $r^2 = x^2 + y^2$;

hence $r = \pm\sqrt{x^2+y^2}$.

Divide the second equation by the first,

$$\tan\theta = \frac{y}{x};\quad \text{hence}\quad \theta = \tan^{-1}\frac{y}{x}.$$

2. Solve for α and β:

$$a\sin\alpha + b\sin\beta = c,$$
$$d\sin\alpha + e\sin\beta = f.$$

Solve for $\sin\alpha$ and $\sin\beta$ as unknowns; hence get α and β.

Exercise. Carry out the solution of example 2. Is the solution possible for all values of $a, b, \ldots, f$? (**62.**)

3. Solve for r and θ:

$$ar\sin\theta + br\cos\theta = c,$$
$$a'r\sin\theta + b'r\cos\theta = c'.$$

Solve for $\gamma\sin\theta$ and $\gamma\cos\theta$ as unknown; then proceed as in example 1.

Exercise. Carry out the solution in example 3.

4. Solve for x and y:

$$y = \sin x,$$
$$y = \sin 2x.$$

Subtracting, $\sin 2x - \sin x = 0$ or $2\sin x\cos x - \sin x = 0$.

Hence $\sin x = 0$ or $\cos x = \frac{1}{2}$.

$$\therefore\quad x = \pm n\pi \quad\text{or}\quad \pm 60^\circ \pm 2n\pi.$$
$$\therefore\quad y = 0 \quad\text{or}\quad \pm\tfrac{1}{2}\sqrt{3}.$$

Exercise. Solve example 4 graphically.

5. Solve for y and t:

$$y = a\sin(nt + b),$$
$$y = a'\sin(nt + b').$$

Equating the values of y, and expanding,

$$a(\sin nt\cos b + \cos nt\sin b) = a'(\sin nt\cos b' + \cos nt\sin b').$$

Dividing by $\cos nt$ and solving for $\tan nt$,

$$\tan nt = \frac{a'\sin b' - a\sin b}{a\cos b - a'\cos b'}.$$

This determines a set of values of nt. Then y is obtained by substituting in either of the given equations.

6. Solve for r, θ, and ϕ:

$$x = r\cos\theta\cos\phi,$$
$$y = r\cos\theta\sin\phi,$$
$$z = r\sin\theta.$$

Dividing the second equation by the first, we have

$$\frac{y}{x} = \tan\phi; \quad \text{hence} \quad \phi = \tan^{-1}\frac{y}{x}.$$

Squaring the first two equations and adding,

$$x^2 + y^2 = r^2\cos^2\theta; \quad \text{hence} \quad r\cos\theta = \pm\sqrt{x^2+y^2}.$$

Combining this result with the third equation, as in example 1, we have

$$\tan\theta = \frac{z}{\pm\sqrt{x^2+y^2}}; \quad \text{hence} \quad \theta = \tan^{-1}\frac{z}{\pm\sqrt{x^2+y^2}}.$$
$$r^2 = x^2 + y^2 + z^2.$$

168. Exercises.

Solve for r and θ:

1. $r\cos\theta = 3$, $r\sin\theta = 4$.

2. $r\cos\theta = 12$, $r\sin\theta = -5$.

3. $r\cos\theta = -9$, $r\sin\theta = -40$.

4. $r\cos\theta + 2\gamma\sin\theta = 3$, $r\sin\theta = 1$.

5. $r(2\sin\theta + 3\cos\theta) = 1$, $r(\sin\theta + 4\cos\theta) = 1$.

6. $r\sin\left(\theta + \frac{\pi}{4}\right) = 2$, $r\cos\left(\theta - \frac{\pi}{4}\right) = 1$.

7. $r = \sin\left(\theta + \frac{\pi}{4}\right)$, $2r = \sin\left(\theta - \frac{\pi}{4}\right)$.

8. $r = 2\sin\left(2\theta - \frac{\pi}{3}\right)$, $r = 3\sin\left(\theta + \frac{2\pi}{3}\right)$.

Solve for r, θ, and ϕ:

9. $r\cos\theta\cos\phi = 3$, $r\cos\theta\sin\phi = 4$, $r\sin\theta = 5$.

10. $r\cos\theta\cos\phi = -1$, $r\cos\theta\sin\phi = 1$, $r\sin\theta = -2$.

Eliminate θ from the following equations:

11. $x = r\cos\theta$; $y = r\sin\theta$.

12. $x = a\cos\theta$; $y = b\sin\theta$.

13. $x = a^3\cos^3\theta$; $y = b^3\sin^3\theta$.

14. $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$; $\frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta = -1$.

15. Eliminate θ and ϕ from the equations

$$x = r\cos\theta\cos\phi; \quad y = r\cos\theta\sin\phi; \quad z = r\sin\theta.$$

16. The same for the equations

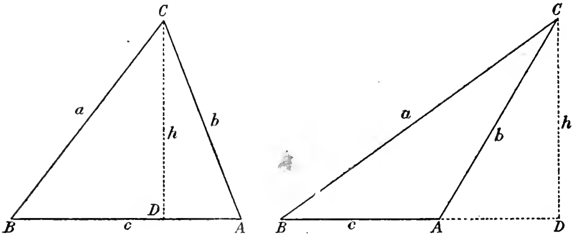
$$x = a\cos\theta\cos\phi; \quad y = b\cos\theta\sin\phi; \quad z = c\sin\theta.$$

CHAPTER X

Oblique Plane Triangles

169. Between the six parts of a plane triangle there exist, aside from the angle-sum equal to 180°, two other fundamental relations which we proceed to obtain. Additional relations will then be derived from these.

The Law of Sines. — *In any plane triangle, the sides are proportional to the sines of the opposite angles.*



Let ABC be the triangle, CD one of its altitudes. Two cases arise, according as D falls within or without the base (figures). Then in the first figure,

from $\triangle ACD$, $\qquad h = b \sin A;$
from $\triangle BCD$, $\qquad h = a \sin B;$

equating the values of h,

$$b \sin A = a \sin B, \text{ or } a : b = \sin A : \sin B.$$

In the second figure,

from $\triangle ACD$, $\qquad h = b \sin(\pi - A) = b \sin A;$
from $\triangle BCD$, $\qquad h = a \sin B;$

equating the values of h, we find the same result as before.

By drawing perpendiculars from the other vertices and combining results we have the **Law of Sines,**

(1) $$a : b : c = \sin A : \sin B : \sin C.$$

170. The Law of Cosines. — *In any plane triangle, the square of any side equals the sum of the squares of the other two sides, minus twice their product by the cosine of their included angle.*

In the above figures let $\overline{AD} = m$. Then

	First figure.	*Second figure.*
in $\triangle ACD$,	$m=b\cos A;$	$m=b\cos(\pi-A)=-b\cos A;$
in $\triangle BCD$,	$a^2=h^2+(c-m)^2$	$a^2=h^2+(c+m)^2$
	$=h^2+c^2-2cm+m^2$	$=h^2+c^2+2cm+m^2$
	$=b^2+c^2-2cm.$	$=b^2+c^2+2cm.$

Replacing m by its value above, we have in either case,

(2) $$a^2 = b^2 + c^2 - 2bc\cos A.$$

(2′) Similarly, $$b^2 = a^2 + c^2 - 2ac\cos B.$$

(2″) $$c^2 = a^2 + b^2 - 2ab\cos C.$$

171. The Law of Tangents. — *In any plane triangle, the difference of two sides is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.*

From the law of sines we have,

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

By composition and division, and subsequent reduction we have,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}$$

$$= \cot\tfrac{1}{2}(A+B)\tan\tfrac{1}{2}(A-B).$$

That is,

(3) $$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

Similarly,

(3′) $$\frac{a-c}{a+c}=\frac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)},$$

(3″) $$\frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$$

The symmetry of these formulas makes them easy to remember. In actual practice, they are used in slightly modified form. Thus the first of them is written,

$$\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\tan\frac{1}{2}(A+B).$$

Similarly for the other two.

172. Functions of the Half-Angles. — When the three sides of a triangle are known, its angles are best calculated by the formulas now to be derived.

From the law of cosines we have,

$$\cos A=\frac{b^2+c^2-a^2}{2\,bc}.$$

In practice this formula is not convenient unless a, b, and c happen to be small numbers. Now

$$\sin\frac{1}{2}A=\sqrt{\frac{1-\cos A}{2}}.\quad\left(\text{Why not }\pm\sqrt{\frac{1-\cos A}{2}}?\right)$$

But $$1-\cos A=1-\frac{b^2+c^2-a^2}{2\,bc}$$

$$=\frac{2\,bc-b^2-c^2+a^2}{2\,bc}$$

$$=\frac{a^2-(b-c)^2}{2\,bc}$$

$$=\frac{(a+b-c)\,(a-b+c)}{2\,bc}.$$

Let $2\,s=a+b+c$, or $s=\frac{1}{2}(a+b+c)$.

Then $a+b-c=2\,(s-c)$, and $a-b+c=2\,(s-b)$.

Then $$1-\cos A=\frac{4\,(s-b)\,(s-c)}{2\,bc},$$

and

(4) $$\sin\frac{1}{2}A=\sqrt{\frac{(s-b)\,(s-c)}{bc}}.$$

Similarly,

(4′) $$\sin \frac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{ac}}.$$

(4″) $$\sin \frac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

Observe that the sides appearing explicitly under the radical *include* the angle to be calculated.

To obtain $\cos \frac{1}{2} A$, we have

$$\cos \frac{1}{2} A = \sqrt{\frac{1 + \cos A}{2}}.$$

But $$1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2\,bc}$$

$$= \frac{(b+c)^2 - a^2}{2\,bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2\,bc}$$

$$= \frac{4\,s\,(s-a)}{2\,bc}.$$

Hence

(5) $$\cos \frac{1}{2} A = \sqrt{\frac{s\,(s-a)}{bc}}.$$

Similarly,

(5′) $$\cos \frac{1}{2} B = \sqrt{\frac{s\,(s-b)}{ac}}.$$

(5″) $$\cos \frac{1}{2} C = \sqrt{\frac{s\,(s-c)}{ab}}.$$

Dividing sine by cosine we have

(6) $$\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s\,(s-a)}}.$$

(6′) $$\tan \frac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{s\,(s-b)}}.$$

(6″) $$\tan \frac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s\,(s-c)}}.$$

If $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

then

$$\tan \frac{1}{2} A = \frac{r}{s - a}, \tag{7}$$

$$\tan \frac{1}{2} B = \frac{r}{s - b}, \tag{7'}$$

$$\tan \frac{1}{2} C = \frac{r}{s - c}. \tag{7''}$$

All these formulas for the half-angle should be memorized, preferably in *verbal form,* so that a single statement contains all three formulas of any one set.

173. Solution of Plane Oblique Triangles. — A triangle is determined, except in such cases as will be specially mentioned, when three parts are given, of which one at least must be a side. The calculation of the other parts is called "solving the triangle."

Four cases arise, according to the nature of the given parts.

I. *Given two angles and one side.*

II. *Given two sides and their included angle.*

III. *Given two sides and an opposite angle.*

IV. *Given three sides.*

The method for treating each case will now be considered.

174. Case I. Given two angles and one side, as A, B, a.

Formulas for finding the other parts, C, b, c.

$$C = 180° - (A + B).$$

From the law of sines,

$$b = a \frac{\sin B}{\sin A}; \quad c = a \frac{\sin C}{\sin A}.$$

Check. It is important to have a check on the accuracy of the calculated parts. For this purpose use any formula involving as many as possible of these parts.

In this case we use

$$\frac{b}{c} = \frac{\sin B}{\sin C}, \quad \text{or} \quad b \sin C = c \sin B.$$

Example. Given $A = 50°, B = 60°, a = 150.$

To find C, b, and c.

Solution by Natural Functions.

$$C = 180° - (50° + 60°) = 70°.$$

$$b = a\frac{\sin B}{\sin A} = \frac{150 \times .8660}{.7660} = \mathbf{169.58.}$$

$$c = a\frac{\sin C}{\sin A} = \frac{150 \times .9397}{.7660} = \mathbf{184.01.}$$

Check. $$b \sin C = c \sin B,$$

or $$169.58 \times .9397 = 184.01 \times .8660,$$

or $$159.35 = 159.35.$$

Logarithmic Solution.

$$C = 180 - (A + B).$$

$$b = a\frac{\sin B}{\sin A}; \quad \log b = \log a + \log \sin B + \text{colog} \sin A.$$

$$c = a\frac{\sin C}{\sin A}; \quad \log c = \log a + \log \sin C + \text{colog} \sin A.$$

Check. $b \sin C = c \sin B$; $\log b + \log \sin C = \log c + \log \sin B$.

We now write out the following scheme:

	$A + B =$	$C = 180° - (A + B) =$	
	$\log a =$	$\log a =$	
	$\log \sin B =$	$\log \sin C =$	
	colog $\sin A =$ ____	colog $\sin A =$ ____	
	$\log b =$	$\log c =$	
	$b =$	$c =$	
Check.	$\log b =$	$\log c =$	
	$\log \sin C =$ ____	$\log \sin B =$ ____	

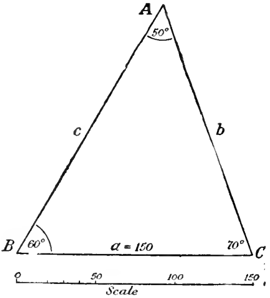
Now turn to the tables and take out all the logarithms required, inserting them in their proper places. Add to obtain log b and log c. Insert these in the check and add. If the sums in the check agree, or differ by only a unit in the last figure, the numerical work is correct. Then look up b and c.

On making these calculations with the data in our example the scheme appears as below.

	$A + B = 110°.$	$C = 180° - 110° = 70°$
	$\log a = 2.1761$	$\log a = 2.1761$
	$\log \sin B = 9.9375$	$\log \sin C = 9.9730$
	$\text{colog} \sin A = \underline{0.1157}$	$\text{colog} \sin A = \underline{0.1157}$
	$\log b = 2.2293$	$\log c = 2.2648$
	$b =$ **169.6**	$c =$ **184.0**
Check.	$\log b = 2.2293$	$\log c = 2.2648$
	$\log \sin C = \underline{9.9730}$	$\log \sin B = \underline{9.9375}$
	2.2023	2.2023

Remark. In calculating with four-place logarithms, three significant figures of the resulting numbers are usually correct. The fourth figure should be retained, but may be one or more units in error. It is rarely worth while to retain more than four significant figures.

A similar remark applies to 5-, 6-, and 7-place tables. See chapter on numerical computation.



Graphic Solution of Case I; given A, B, and a.

Calculate $C = 180° - (A + B)$. Lay off a line segment equal to a and at its extremities construct angles B and C, prolonging their free sides until they meet at A (figure). Scale off the lengths of b and c. The figure shows the triangle already solved above. From it we have

$$b = 167, \ c = 181.$$

No solution is possible when $A + B > 180°$.

Exercises. Solve the following triangles, including graphic solutions.

1. $A = 55°$	$B = 72°$	$a = 1000.$
2. $A = 65° \, 25'$	$B = 78° \, 23'$	$a = 4.245.$
3. $C = 34° \, 48'$	$A = 100° \, 17'$	$b = 0.5575.$
4. $B = 115° \, 10'.5$	$C = 40° \, 22'.3$	$c = 0.00275.$
5. $B = 88° \, 20'$	$C = 105° \, 30'$	$a = 10.$

175. Case II. Given two sides and the included angle, as *a, b, C*.

To solve the triangle we calculate $\frac{1}{2}(A+B)$ as the complement of $\frac{1}{2}C$; then $\frac{1}{2}(A-B)$ is calculated by formula (3). Angles A and B are then determined and hence all the angles are known. We can then compute c in two ways by means of the law of sines. The agreement of the two values of c furnishes a check on the computations.

Formulas.

$$\tfrac{1}{2}(A+B) = 90^\circ - \tfrac{1}{2}C,$$

$$\tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\tan\tfrac{1}{2}(A+B),$$

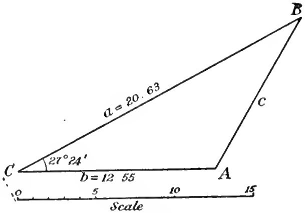
$$c = a\frac{\sin C}{\sin A} = b\frac{\sin C}{\sin B}.$$

Scheme for Logarithmic Solution.

$a =$	$\log(a-b) =$	$\frac{1}{2}(A+B) =$
$b =$	$\operatorname{colog}(a+b) =$	$\frac{1}{2}(A-B) =$
$a+b =$	$\log\tan\frac{1}{2}(A+B) =$	$A =$
$a-b =$	$\log\tan\frac{1}{2}(A-B) =$	$B =$

$\log a =$	$\log b =$
$\log\sin C =$	$\log\sin C =$
$\operatorname{colog}\sin A =$	$\operatorname{colog}\sin B =$
$\log c =$	$\log c =$
$c =$	$c =$

Graphic Solution. Construct angle C and on its sides lay off lengths a and b, starting from the vertex. Complete the triangle, and measure c, A, and B (figure, constructed for example below).



A solution is possible provided $C < 180^\circ$.

Example. Given $b = 12.55$, $a = 20.63$, $C = 27^\circ\,24'$. Solve the triangle.

Logarithmic Solution.

$$\tfrac{1}{2}(A+B) = 90° - \tfrac{1}{2}C = 90° - 13°\ 42' = 76°\ 18'.$$

$a =$	20.63	$\log(a-b) =$	0.9074	$\frac{1}{2}(A+B) =$	76° 18′
$b =$	12.55	$\text{colog}(a+b) =$	8.4792	$\frac{1}{2}(A-B) =$	44° 58′.4
$a+b =$	33.18	$\log\tan\frac{1}{2}(A+B) =$	0.6130	$A =$	**121° 16′.4**
$a-b =$	8.08	$\log\tan\frac{1}{2}(A-B) =$	9.9996	$B =$	**31° 19′.6**

$\log a =$	1.3145	$\log b =$	1.0986
$\log\sin C =$	9.6630	$\log\sin C =$	9.6630
$\text{colog}\sin A =$	0.0682	$\text{colog}\sin B =$	0.2841
$\log c =$	1.0457	$\log c =$	1.0457
$c =$	**11.11**	$c =$	11.11

Graphic Solution. This is shown in the figure above. Let the student scale off the known parts.

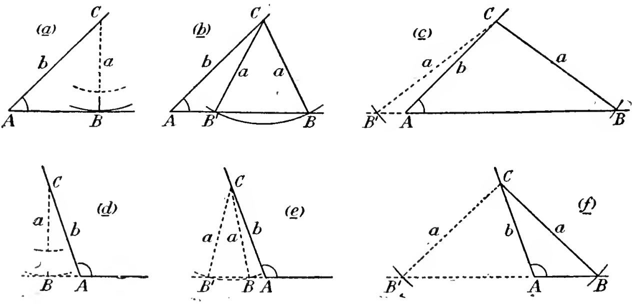
Exercises. Solve the following triangles:

1. $a = 1500$, $b = 750$, $C = 58°$.
2. $b = 15.25$, $c = 12.65$, $A = 98°\ 40'$.
3. $a = 1.002$, $b = 0.8656$, $C = 130°\ 48'$.
4. $b = 6238$, $c = 4812$, $A = 75°\ 22'$.
5. $a = 16.21$, $c = 22.48$, $B = 36°\ 54'$.

176. Case III. Given two sides and an opposite angle, as *a*, *b*, *A*.

This is known as the *ambiguous case.* We begin by studying the graphic solution.

Lay off angle A and on one of its sides take $AC = b$. With C as center and radius equal to a, strike an arc of a circle. The figures show the various possibilities arising in the construction, the first three for $A < 90°$, the last three for $A > 90°$.



In each case the perpendicular from C on the other side of angle A is equal to $b \sin A$. Inspection of the figures then shows that

when $A < 90°$ and $a < b \sin A$, no triangle is possible;

when $A < 90°$ and $a = b \sin A$, a right triangle results;

when $A < 90°$ and $b > a > b \sin A$, two oblique triangles result;

when $A < 90°$ and $a > b$, one oblique triangle results;

when $A > 90°$ and $a \leqq b$, no solution is possible;

when $A > 90°$ and $a > b$, one oblique triangle results.

It is always possible therefore to state in advance what the nature of the solution in a given case will be.

Formulas. Given a, b, A.

$$\begin{cases} \sin B = \dfrac{b}{a} \sin A. \\ B' = 180° - B. \end{cases} \quad \begin{cases} C = 180° - (A + B). \\ C' = 180° - (A + B'). \end{cases} \quad \begin{cases} c = a \dfrac{\sin C}{\sin A} = b \dfrac{\sin C}{\sin B}. \\ c' = a \dfrac{\sin C'}{\sin A} = b \dfrac{\sin C'}{\sin B'}. \end{cases}$$

Check. The agreement of the values of c and c' as calculated from the two expressions for each of them furnishes a partial check on the calculations. It does not guard against an error in log sin C, which may be checked independently. A complete check is furnished by (6) of (**172**).

In carrying out the calculations according to the formulas above, the various cases shown in the figures are indicated as follows:

(a) $\log \sin B \geqq 0$; no solution, or right triangle.

(b) retain both B and B'; two solutions.

(c) $A + B' > 180°$, hence reject B'; one solution.

(d) $\log \sin B \geqq 0$; no solution.

(e) $A + B > 180°$ and $A + B' > 180°$; no solution.

(f) As in (c); one solution.

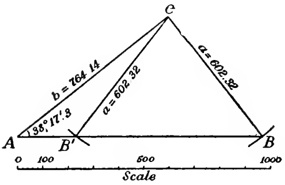
In a given numerical example the nature of the solution always becomes apparent during the progress of the computations.

Example. Given $a = 602.3$, $b = 764.1$, $A = 38° 17'.3$.

Logarithmic Solution.*

$\log b = 2.88316$	$\log a = 2.77981$	$\log b = 2.88316$
$\operatorname{colog} a = 7.22019$	$\log \sin C = 0.00000$	$\log \sin C = 0.00000$
$\log \sin A = 9.79217$	$\operatorname{colog} \sin A = 0.20783$	$\operatorname{colog} \sin B = 0.10448$
$\log \sin B = 9.89552$	$\log c = 2.98764$	$\log c = 2.98764$
$B = \mathbf{51° 50'.0}$	$c = \mathbf{971.9}$	
$B' = \mathbf{128° 10'.0}$	$\log a = 2.77981$	$\log b = 2.88316$
$A + B = 90° \; 7'.3$	$\log \sin C' = 9.36960$	$\log \sin C' = 9.36960$
$A + B' = 166° 27'.3$	$\operatorname{colog} \sin A = 0.20783$	$\operatorname{colog} \sin B' = 0.10448$
$C = \mathbf{89° 52'.7}$	$\log c' = 2.35724$	$\log c' = 2.35724$
$C' = \mathbf{13° 32'.7}$	$c' = \mathbf{227.6}$	

Graphic Solution. This is shown in the figure, from which the unknown parts may be scaled off.



Exercises. Solve the triangles whose given parts are:

1. $a = 29.95$, $b = 37.17$, $A = 42° 24'$.
2. $a = 1756$, $b = 745$, $A = 67° 30'$.
3. $b = .728$, $c = .542$, $B = 105° 44'$.
4. $b = 6.174$, $a = 2.614$, $B = 32° 22'$.

177. Case IV. Given the three sides, *a*, *b*, *c*.

The angles may be calculated from either the sine, cosine, or tangent of the half-angles. When all three angles are wanted, it is best to use the tangent. There is no solution when one side equals or exceeds the sum of the other two.

Formulas.

$$s = \frac{1}{2}(a + b + c); \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

$$\tan\frac{1}{2}A = \frac{r}{s-a}; \quad \tan\frac{1}{2}B = \frac{r}{s-b}; \quad \tan\frac{1}{2}C = \frac{r}{s-c}.$$

Check. $\frac{1}{2}(A + B + C) = 90°$.

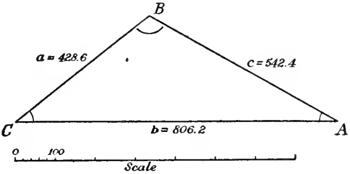
* The fifth figure is carried to avoid accumulation of error. This is advisable if all possible accuracy is desired.

Example. Given $a = 428.6$, $b = 806.2$, $c = 542.4$.

Logarithmic Solution.

a	428.6	colog s	7.0513	$\frac{1}{2}A$	14° 47′.7
b	806.2	$\log(s-a)$	2.6628	$\frac{1}{2}B$	55° 51′.5
c	542.4	$\log(s-b)$	1.9159	$\frac{1}{2}C$	19° 20′.5
$2s$	1777.2	$\log(s-c)$	2.5393	Check	89° 59′.7
s	888.6	2	4.1693	A	29° 35′.4
$s-a$	460.0	$\log r$	2.0846	B	111° 43′.0
$s-b$	82.4	$\log\tan\frac{1}{2}A$	9.4218	C	38° 41′.0
$s-c$	346.2	$\log\tan\frac{1}{2}B$	0.1687		179° 59′.4
Check	1777.2	$\log\tan\frac{1}{2}C$	9.5453		

Graphic Solution. This is shown in the figure. By measuring we find $A = 29°$, $B = 112°$, $C = 38°$.



Exercises. Solve the triangles whose given parts are:

1. $a = 6192$, $b = 4223$, $c = 7415$.
2. $a = 156.21$, $b = 300.15$, $c = 410.32$.
3. $a = 0.00245$, $b = 0.00405$, $c = 0.00536$.
4. $a = 52.76$, $b = 22.84$, $c = 28.41$.

178. Areas of Oblique Plane Triangles.—Referring to the figures of (**169**), we see that h is the altitude drawn on side c as base. Hence if K denote the area of the triangle, we have

$$(8) \qquad K = \tfrac{1}{2}hc = \tfrac{1}{2}ac \sin B. \qquad (h = a \sin b.)$$

Hence, the area of a plane triangle equals half the product of two sides by the sine of their included angle.

The area is also expressible in simple form in terms of the sides. In the formula above replace $\sin B$ by $2 \sin \frac{1}{2} B \cos \frac{1}{2} B$. Then

$$K = ac \sin \tfrac{1}{2} B \cos \tfrac{1}{2} B$$
$$= ac\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{s(s-b)}{ac}},$$

by (4′) and (5′) of (**172**). Hence,

(9) $$K = \sqrt{s(s-a)(s-b)(s-c)}.$$

When the given parts of the triangle are such that neither of the above formulas applies directly, it is usually best to calculate additional parts so that one of these formulas may be used.

179. Exercises and Problems.

1.	**2.**	**3.**	**4.**
$a = 183.9$,	$a = 1.925$,	$a = 42.31$,	$a = .41409$,
$b = 584.9$,	$b = 2.243$,	$b = 71.70$,	$b = .49935$,
$c = 166.6$.	$c = 7.25$.	$c = 71.35$.	$c = .18182$.
5.	**6.**	**7.**	**8.**
$a = 183.7$,	$a = 283.6$,	$a = 783$,	$c = 22.504$,
$A = 36° 55'.9$,	$A = 11° 15'$,	$B = 42° 27'$,	$B = 55° 11'$,
$C = 70° 58'.2$.	$B = 47° 12'$.	$C = 55° 41'$.	$C = 45° 34'$.
9.	**10.**	**11.**	**12.**
$b = 3069$,	$b = 100.2$,	$a = 3186$,	$a = .8712$,
$B = 15° 51'$,	$B = 48° 59'$,	$b = 17156$,	$b = .4812$,
$A = 58° 10'$.	$C = 76° 3'$.	$A = 147° 12'$.	$A = 24° 31'$.
13.	**14.**	**15.**	**16.**
$a = 1523$,	$A = 61° 16'$,	$a = .39363$,	$b = 147.26$,
$b = 1891$,	$a = 95.12$,	$c = .23655$,	$c = 109.71$,
$A = 21° 21'$.	$b = 127.52$.	$C = 22° 32'$.	$A = 41° 15'$.
17.	**18.**	**19.**	**20.**
$b = .5863$,	$a = 10.374$,	$b = 6.4082$,	$b = .8869$,
$a = .8073$,	$c = 9.998$,	$c = 18.406$,	$a = 3.0285$,
$C = 58° 47'$.	$B = 49° 50'$.	$A = 33° 31'$.	$C = 128° 7'$.
21.	**22.**	**23.**	**24.**
$a = .8706$,	$a = 20.71$,	$A = 41° 13'$,	$a = 4663$,
$b = .0916$,	$b = 18.87$,	$a = 77.04$,	$b = 4075$,
$c = .7902$.	$C = 55° 12' 3''$.	$b = 91.06$.	$C = 58°$.
25.	**26.**	**27.**	**28.**
$a = 43031$,	$a = 16082$,	$a = .00502$,	$b = 2584$,
$c = 31788$,	$c = 13542$,	$b = .00558$,	$c = 5726$,
$A = 19° 12'.7$.	$C = 52° 24'.3$.	$c = .00466$.	$A = 27° 13'$.
29.	**30.**	**31.**	**32.**
$b = 37403$,	$a = 6148$,	$a = .01520$,	$b = 8204$,
$a = 49369$,	$c = 7512$,	$b = .03366$,	$c = 9098$,
$A = 81° 47'$.	$A = 133° 30'$.	$c = .02114$.	$A = 62° 9'.6$.

33.	34.	35.	36.
$a = 532$,	$a = 290$,	$a = .000299$,	$c = 7025$,
$b = 704$,	$c = 356$,	$c = .000180$,	$b = 8530$,
$C = 73°$.	$C = 41° 10'$.	$A = 63° 50'$.	$C = 40°$.

37.	38.	39.	40.
$b = 1482$,	$a = .2785$,	$B = 50° 20' 54''$,	$C = 49° 47' 26''$.
$a = 1284$,	$b = .2275$,	$a = 235.64$,	$c = 725.52$,
$A = 27° 18'$.	$B = 65° 40'$.	$b = 284.31$.	$b = 950.04$.

In any triangle ABC, whose sides, opposite angles A, B, C, respectively, are a, b, c, show that:

41. $b(s-b)\cos^2\frac{B}{2} = a(s-a)\cos^2\frac{A}{2}$.

42. $a = b\cos C + c\cos B$.

43. $(a-b)(1+\cos C) = c(\cos B - \cos A)$.

44. $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$.

45. $(b+c-a)\tan\frac{A}{2} = (c+a-b)\tan\frac{B}{2}$.

46. $(b+c)(1-\cos A) = a(\cos B + \cos C)$.

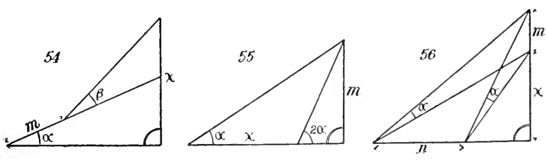
47. $(a^2-b^2+c^2)\tan B = (a^2+b^2-c^2)\tan C$.

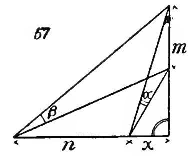
48. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

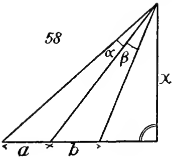
49. The radius of the inscribed circle is $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

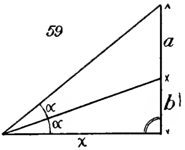
50. The diameter of the circumscribed circle is $a \csc A$.

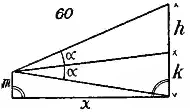
Calculate x in terms of the other quantities in each figure below, where a right angle is indicated by a double arc; in each case find the value of x for an assumed set of values of the literal quantities:

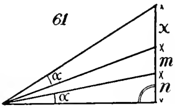


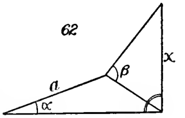












63. Find the lengths of diagonals and the area of a parallelogram two of whose sides are 5 ft. and 8 ft., their included angle being 60°.

64. Two sides of a parallelogram are a and b, their included angle C; show that the area is $ab \sin C$.

65. The sides of a triangle are 4527, 7861, 6448; find the length of the median drawn to the shortest side.

66. The sides of a triangle are in the ratio of 2 : 3 : 4; find the cosine of the smallest angle.

67. The angles of a triangle are as 3 : 4 : 5; the shortest side is 500 ft.; solve the triangle.

68. The angles of a triangle are as 1 : 2 : 3; the longest side is 100 ft.; solve the triangle.

69. From a station on level ground due south of a hill, the angle of elevation of the top is 15°; from a point 2000 ft. east of this station the angle of elevation is 12°; how high is the hill?

70. The angle of elevation of the top of a building 100 ft. high is 60°; what will be the angle at double the distance?

71. A flag-pole on a building subtends an angle of 7° 40′ at a point on the ground 500 ft. from the building; on approaching 100 ft., the pole subtends an angle of 7° 50′; find the height of the pole and the building.

72. On level ground, 250 ft. from the foot of a building, the angles of elevation of the top and bottom of a flag-pole surmounting the building are 38° 43′ and 31° 2′ respectively; find the height of the building and the pole.

73. From level ground the angle of elevation of the top of a hill is 11° 30′; after approaching 3000 ft. up an incline of 3° 27′, the angle of elevation of the top is 21° 32′; how high is the hill?

74. From a level plain, the angle of elevation of a distant mountain top is 5° 50′; after approaching 4 miles, the angle is 8° 40′; how high is the mountain?

75. From a point 60 ft. above sea level the angle between a distant ship and the sea horizon (the offing) is 20′; how far away is the ship? [Consider the surface of the sea as a plane, and the distance to the horizon 10 miles. See (**226**) ex. (4).]

76. From a point on level ground the angle of elevation of the top of a hill is 14° 12′; on approaching 1000 ft., the angle is 17° 50′; how high is the hill?

77. A building surmounted by a flag-pole 20 ft. high stands on level ground. From a point on the ground the angles of elevation of the top and the bottom of the pole are 53° 5′ and 45° 11′ respectively. How high is the building?

78. On approaching 1 mile toward a hill, the angle of elevation of its top is doubled; on approaching another mile, the angle is again doubled; how high is the hill?

79. A and B are two points neither of which is visible from the other. To determine the distance AB, two stations C and D are chosen and the following measurements made: $CD = 500$ ft.; $\angle ACD = 30° 25' 15''$; $\angle ACB = 85° 40' 20''$; $\angle BDC = 35° 14' 50''$; $\angle BDA = 80° 20' 25''$; find AB.

80. In a chain of three non-overlapping triangles, the following data are known:

$$AB = 1000 \text{ ft.}$$

$\triangle ABC$,	$\triangle ACD$,	$\angle CDE$,
$\angle A = 44° 36'$,	$\angle A = 56° 32'$,	$\angle C = 55° 30'$,
$\angle C = 40° \ 0'$;	$\angle C = 50° 20'$;	$\angle E = 77° 02'$;

Calculate DE. (Express DE in terms of AB and the necessary angles by the law of sines.)

81. In a chain of four non-overlapping triangles, the following data are known:

$$AB = 11289 \text{ meters.}$$

$\triangle ABC$,	$\triangle CBD$,	$\triangle DBE$,	$\triangle DEF$,
$\angle A = 58° 10' 35''$,	$\angle B = 86° 50' 0''$,	$\angle D = 79° 12' 8''$,	$\angle D = 50° 41' 5''$,
$\angle B = 69° 55' 0''$;	$\angle C = 46° 48' 0''$;	$\angle B = 73° 29' 10''$;	$\angle E = 45° 20' 40''$;

calculate EF.

82. In a chain of five consecutive triangles, each having a side in common with the preceding, as ABC, CBD, BDE, DEF, EFG, express FG in terms of AB and the necessary angles.

83. A tower 50 ft. high stands on the edge of a cliff 150 ft. high. At what distance from the foot of the cliff will the tower subtend an angle of 5°?

84. The sides of a triangle are 100, 150, 200 ft. At the vertex of the smallest angle a line 100 ft. long is drawn perpendicular to the plane of the triangle. Find the angles subtended at the farther end of this line by the sides of the triangle.

85. A right triangle whose perimeter is 100 ft. rests with its hypotenuse on a plane, the vertex of the right angle being 10 ft. from the plane. The angle between the plane of the triangle and the supporting plane is 30°. Find the sides of the triangle.

86. An equilateral triangle 50 ft. on a side rests with one side on a plane with which its plane makes an angle of 60°. How far is the third vertex from the plane?

87. As in exercise 86, if the triangle, instead of being equilateral, has sides 40, 20, 30 ft. and rests on the shortest side. *Ans.* $\frac{45\sqrt{5}}{4}$.

88. The sides of a triangle are as 1 : 2 : 3, and the longest median is 10 ft. Find the sides and angles.

89. The following measurements of a field $ABCD$ are made: A to B, due north, 10 chains; B to C, N 30° E, 6 chains; C to D, due east, 8 chains; calculate AD, and the area of the field in acres. (1 chain = 4 rods.)

90. The following measurements of a field $ABCDE$ are made: A to B, due east, 25.52 chains; B to C, E 40° 26′ N, 22.25 chains; C to D, N 48° 26′ W, 33.75 chains; D to E, W 31° 15′ S, 18.32 chains; calculate EA and the area of the field in acres.

91. In the field of exercise 89 how much area is cut off by a line due east through B?

92. In the field of exercise 90 where should an east and west line be drawn so as to bisect the area?

93. In the field of exercise 90 where should a north and south line be drawn to cut off 30 acres from the western part of the area?

94. If P be the pull required to move a weight W up a plane inclined to the horizontal at an angle i, and μ the coefficient of friction, then

$$P = W\frac{\sin i + \mu \cos i}{\cos i - \mu \sin i}.$$

Calculate P when $W = 1000$ lbs., $i = 30°$, $\mu = 0.1$.

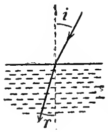
95. In exercise 94, what is i if $P = \frac{1}{2}W$ and $\mu = 0.1$?

96. If l be the length of a plane inclined to the horizontal at an angle i, μ the coefficient of friction and g the acceleration due to gravity (32.+ ft. per sec. per sec.) the time in seconds required by a body to slide down the plane is

$$T = \sqrt{\frac{2l}{g(\sin i - \mu \cos i)}}.$$

What is T when $l = 25$ ft., $i = 20°$, $\mu = 0.1$?

97. In exercise 96, find i when $l = 100$ ft., $\mu = 0.1$, $T = 5$ sec.



98. When light passes from a rarer to a denser medium, the index of refraction μ is determined by the equation

$$\mu = \frac{\sin i}{\sin r}.$$

When $\mu = 1.2$, what must be i (angle of incidence) to give a deflection of 10°?

99. Find the total deflection of a ray which passes through a wedge whose angle is 30° and index of refraction 1.4, if the ray enters the wedge so that the angle of incidence is 25°, and moves in a plane ⊥ to the edge of the wedge.

100. Solve exercise 99 when the angle of the wedge is α, the angle of incidence i, and the index of refraction μ.

CHAPTER XI

The Progressions. Interest and Annuities

180. Arithmetic Progressions. — Let $a, b, c, \ldots, k, l$ be quantities such that the *difference* between any one of them and the preceding one is constant. Then the quantities are said to form an **arithmetic progression.** (We shall abbreviate this into A. P.)

The quantities $a, b, c \ldots, k, l$ are called the *terms* of the progression, a and l the *extremes*, and $b, c, \ldots, k$ the *means*. The constant difference between consecutive terms is called the *common difference.*

Let a denote the first term,
l denote the last term,
d denote the common difference,
n denote the number of terms,
S denote the sum of the terms of any A. P. Then
the second term is $a + d$,
the third term is $a + 2d$,
. . . . .
the last or nth term is $a + (n - 1)d$; that is,

(1) $$l = a + (n-1)d.$$

Also

$$S = a + (a+d) + (a+2d) + \cdots + (a + \overline{n-1}\,d);$$
$$S = l + (l-d) + (l-2d) + \cdots + (l - \overline{n-1}\,d).$$

Adding,

$$2S = (a+l) + (a+l) + \ldots + (a+l) = n(a+l).$$

Hence

(2) $$S = \frac{n}{2}(a+l).$$

Putting for l its value from (1),

(2′) $$S = n\left(a + \frac{n-1}{2}d\right).$$

We shall refer to the five quantities a, l, d, n, S, as the *elements* of the A. P. When any three elements are given, the other two may be found by use of the preceding formulas.

181. *Problem. To insert m arithmetic means between two given quantities, a and l.*

Since there are 2 extremes and m means, the total number of terms is $m + 2$. Hence if d be the common difference,

$$l = a + (m + 2 - 1)\,d;$$

hence

$$d = \frac{l - a}{m + 1}.$$

Then the required means are

$$a + d,\ a + 2\,d,\ \ldots,\ a + md.$$

When $m = 1$ we have only a single mean, called **the arithmetic mean.** It equals $\frac{1}{2}(a + l)$.

182. *Examples.*

1. Find the sum of all the integers from 1 to 100 inclusive.

Here $S = 1 + 2 + 3 + \cdots + 100.$

Then $a = 1,\ l = 100,\ n = 100,$

and $S = \frac{n}{2}(a + l) = \frac{1}{2} \times 100\,(1 + 100) = 5050.$

2. How many terms of the progression 3, 0, -3, . . . are required to make the sum equal -27.

Here $a = 3$, $d = -3$, $S = -27$; to find n.

From (2′), $-27 = n\left(3 - \frac{n-1}{2} \times 3\right)$, or $n^2 - 3\,n - 18 = 0.$

Hence $n = 6$ or $-3.$

Since n must be positive we discard the second value.

3. Find four numbers in A.P., such that the sum of the first and last shall be 12 and the product of the middle two 32.

Let the numbers be $a - 3\,d$, $a - d$, $a + d$, and $a + 3\,d$, with a common difference $2\,d$.

Then $a - 3\,d + a + 3\,d = 12$

and $(a - d)\,(a + d) = 32.$

Hence $a = 6$ and $d = \pm 2.$

Therefore the numbers are

0, 4, 8, 12, or 12, 8, 4, 0.

183. Exercises. Find the last term and the sum of each of the following arithmetic progressions:

1. 7, 11, 15, . . . , to 13 terms;
2. 5, 8, 11, . . . , to 12 terms;
3. 2, $2\frac{1}{2}$, 3, . . . , to 25 terms;
4. 1, 1.1, 1.2, . . . , to 200 terms;
5. 63, 58, 53, . . . , to 8 terms;
6. $x, x+2y, x+4y, \ldots$, to 10 terms;
7. $p, p-\frac{1}{2}q, p-q, \ldots$, to 20 terms.

Find the other elements of the A.P., given that:

8. $a = 10,\ n = 14,\ S = 1050$;
9. $a = 3,\ n = 50,\ S = 3825$;
10. $a = -45,\ n = 31,\ S = 0$;
11. $l = 21,\ n = 7,\ S = 105$;
12. $l = 49,\ n = 19,\ S = 503\frac{1}{2}$;
13. $l = 148,\ n = 27,\ S = 2241$;
14. $l = -143, n = 33,\ S = -2079$;
15. $n = 21,\ S = 1197,\ d = 4$;
16. $n = 35,\ S = 2485,\ d = 3$;
17. $n = 50,\ S = 425,\ d = \frac{1}{3}$.
18. $n = 33,\ S = -33,\ d = -\frac{3}{4}$;
19. $S = 624,\ a = 9,\ d = 4$;
20. $S = 2877,\ a = 7,\ d = 3$;
21. $S = 623,\ d = 5,\ l = 77$;
22. $S = 682.5,\ d = 1.5,\ l = 45$;
23. $S = 95172,\ d = -7,\ l = 567$.

24. Find the sum of the first 100 odd numbers.

25. Find the sum of the first 50 multiples of 7.

26. A body starting from rest falls 16 ft. during the first second, and in every other second 32 ft. more than during the preceding. How far does the body fall in 12 seconds; how far during the 12th second ?

27. According to the rate of fall in exercise 26, how long will the body take to fall 1600 ft ?

28. A body which is projected vertically upward loses 32 ft. of its initial velocity each second. If the velocity of projection is 320 ft. per second, how high will the body rise ?

29. If 100 apples are laid in a straight line, 3 feet apart, how far must a person walk to carry them one at a time to a basket standing beside the first apple ?

184. Geometric Progressions. — If the numbers $a, b, c, \ldots, k, l$ are such that the *ratio* of any number to the preceding number is constant, the numbers form a **geometric progression.** (We abbreviate by writing G. P.)

The expressions "*terms*," "*means*," "*extremes*," are used here as in the case of A.P. The constant ratio of any term to the preceding is called the *ratio* of the geometric progression.

If a, l, n, and S have the same meaning as in the case of the A. P., and if r denote the ratio of the G. P., the first n terms are,

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}.$$

Hence

(1) $$l = ar^{n-1}.$$

Also $$S = a + ar + ar^2 + \cdots + ar^{n-1}$$

and $$rS = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

Therefore $$rS - S = ar^n - a,$$

or $$(r-1)S = (r^n - 1)a.$$

Hence

(2) $$S = a\frac{r^n - 1}{r - 1} = a\frac{1 - r^n}{1 - r}.$$

Substituting from (1) in (2) we have

(2′) $$S = \frac{rl - a}{r - 1}.$$

When any three of the five elements are given, the other two may be obtained by use of two of the preceding formulas. In some cases this involves the solution of an equation of nth degree or of an exponential equation.

185. *Problem. To insert m geometric means between two given numbers a and l.*

The total number of terms being $m + 2$, we have, if r denote the ratio,

$$l = ar^{m+2-1} \quad \text{or} \quad r = \sqrt[m+1]{\frac{l}{a}}.$$

The required geometric means are then

$$ar, ar^2, \ldots, ar^m.$$

When $m = 1$, the resulting single mean between a and l is $\sqrt{al}$. The square root of the product of two quantities is called their **geometric mean.**

186. *Examples.*

1. Find the sum of the first 10 terms of the G. P. 2, 2^2, 2^3, ...

Here $a = 2,\ r = 2,\ n = 10$; hence $S = 2\dfrac{2^{10} - 1}{2 - 1} = 2046.$

2. How many terms of the G. P. 1, 2, 4, . . . are required to make the sum 63?

Here $a = 1,\ r = 2,\ S = 63$; to find n.

From $S = a\dfrac{r^n - 1}{r - 1}$ we have $63 = \dfrac{2^n - 1}{2 - 1}$; or, $64 = 2^n$.

Hence $n = 6$.

3. Four numbers are in geometric progression. The sum of the first and last is 18, the product of the second and third 32. Find the numbers.

Let the numbers be a, ar, ar^2 and ar^3.

Then

$$(1)\quad a + ar^3 = 18; \qquad (2)\quad a^2r^3 = 32.$$

Multiply (1) by a and in the result replace a^2r^3 by 32.

Then $$a^2 + 32 = 18\,a; \quad \text{hence} \quad a = 16 \text{ or } 2.$$

Substituting the values of a in (2) we find $r = \frac{1}{2}$ or 2. Hence the numbers are

$$16,\ 8,\ 4,\ 2; \text{ or } 2,\ 4,\ 8,\ 16.$$

(We disregard the imaginary values of r.)

187. Exercises. Find the last term and the sum of the terms of the following geometric progressions:

1. $4, 8, 16, \ldots$, to 7 terms.

2. $2, 6, 18, \ldots$, to 9 terms.

3. $1, 4, 16, \ldots$, to 7 terms.

4. $9, 3, 1, \ldots$, to 11 terms.

5. $1, \frac{1}{4}, \frac{1}{16}, \ldots$, to 10 terms.

6. $8, 2, \frac{1}{2}$, to 20 terms.

7. $a,\ a(1+x),\ a(1+x)^2, \ldots$ to 8 terms.

8. $m^3,\ mn,\ m^{-1}n^2, \ldots$, to 9 terms.

9. Insert 3 geometric means between 8 and 10368.

10. Insert 5 geometric means between 2 and 31250.

11. Insert 5 geometric means between 36 and $\frac{4}{81}$.

12. Insert 6 geometric means between 3 and 49152.

13. Insert 4 geometric means between 48 and $\frac{3}{64}$.

14. Insert 5 geometric means between 81 and $\frac{256}{81}$.

Calculate the unknown elements, given:

15. $l = 128$, $r = 2$, $n = 7$.

16. $l = 78125$, $r = 5$, $n = 8$.

17. $l = \frac{2}{27}$, $r = \frac{1}{3}$, $n = 5$.

18. $a = 9$, $l = 2304$, $r = 2$.

19. $a = 2$, $l = 64$, $r = 2$

20. $a = 3$, $l = 192\sqrt{2}$, $r = \sqrt{2}$.

21. $a = 2$, $l = 1458$, $S = 2186$.

22. $a = 1$, $l = 2401$, $S = 2801$.

23. $a = 10$, $l = \frac{5}{16}$, $S = 19\frac{11}{16}$.

24. $a = 3125$, $l = 5$, $S = 3905$.

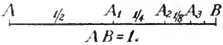
25. $a = 3$, $r = 3$, $S = 29523$.

26. $a = 8$, $r = 2$, $S = 4088$.

27. $r = 2$, $n = 7$, $S = 635$.

28. $l = 1296$, $r = 6$, $S = 1555$.

188. Infinite Geometric Progressions. — Consider a line segment AB of unit length, and bisect it at A_1, then bisect A_1B at A_2, A_2B at A_3 and so on (figure).



The points of bisection $A_1, A_2, A_3, \ldots$ continually approach B and the sum of the segments $AA_1 + A_1A_2 + A_2A_3 + \cdots$ approaches AB or 1. But the sum of these segments is represented numerically by the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots, \quad \text{or} \quad \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots,$$

and hence by taking n large enough we can make the sum

$$S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$$

differ from 1 by as little as we please. Hence we take

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \text{ to infinity} = 1.$$

The sum S_n above is a geometric progression with $r = \frac{1}{2}$ and $a = \frac{1}{2}$. Its sum to n terms is therefore

$$S_n = \frac{1}{2}\,\frac{(\frac{1}{2})^n - 1}{\frac{1}{2} - 1}.$$

As n increases, $(\frac{1}{2})^n$ approaches 0, and S_n approaches the value $\frac{1}{2}\,\frac{0 - 1}{\frac{1}{2} - 1} = 1$, as found above.

A geometric progression in which the number of terms increases without limit is called an *infinite geometric progression.*

For the sum of n terms of any G. P. we have

$$S_n = a\frac{r^n - 1}{r - 1} = a\frac{1 - r^n}{1 - r}.$$

If now $r < 1$, then r^n approaches 0 when n approaches ∞, and the formula for the sum of an infinite G. P. is

$$S = \frac{a}{1 - r}, \quad \text{provided } |r| < 1.$$

(When $r = 1$, or when $r > 1$, S is infinite.)

Example. A ball is thrown vertically upward to a height of 60 ft. On striking the ground it always rebounds to one-third the height from which it fell. How far will it travel ?

The distance covered during the first rise and fall is 120 ft., during the second rise and fall, $\frac{1}{3} \times 120$ ft., during the third, $\frac{1}{3^2} \times 120$ ft., and so on indefinitely. We have an infinite G. P., with $a = 120$ and $r = \frac{1}{3}$. Hence the total distance will be

$$S = \frac{120}{1 - \frac{1}{3}} = 180 \text{ ft.}$$

189. Exercises. Sum the following infinite geometric progressions:

1. $8, 2, \frac{1}{2}, \ldots$
2. $1, \frac{1}{4}, \frac{1}{16}, \ldots$
3. $5, 3, \frac{9}{5}, \ldots$
4. $2, \frac{2}{7}, \frac{2}{49}, \ldots$
5. $1, -\frac{1}{2}, +\frac{1}{4}, -\frac{1}{8}, \ldots$
6. $3, -1, \frac{1}{3}, -\frac{1}{9}, \ldots$

7. If in the example worked above the ball requires 4 seconds for the first rise and fall, and half as much time for any subsequent rise and fall as for the preceding, how long before the ball will come to rest ?

8. How far has the ball in the above example traveled at the 10th rebound ?

190. Harmonic Progressions. — If the numbers $a, b, c, \ldots, k, l$ are such that their reciprocals form an arithmetic progression, they are said to be in harmonic progression (abbreviated to H. P.).

Problems relating to harmonic progressions are solved by reduction to A. P.

If a, b, c form a H. P., then b is called the harmonic mean between a and c. Let the student show that we then have

$$b = \frac{2\,ac}{a+c}.$$

191. Exercises.

1. In an A. P. the sum of the 9th and 12th terms is 40; the difference between the squares of the 15th and 11th terms is 400. Find a and d.

2. In an A. P. of 10 terms, the sum of the terms is 65 and the sum of their squares 1165. Find a and d.

3. In an A. P. of 20 terms, the sum of the 3rd and 12th terms is 30, the product of the two middle terms is 725. Find a and d.

4. In an A. P. of 14 terms, the product of the first and the last is 276 and the product of the middle two is 1326. Find a and d.

5. Find four numbers in A. P. such that their product is 840 and their sum 11.

6. Find four numbers in A. P. such that their product is h and the sum of their squares is k.

7. Find five numbers in A. P. such that their product is a, their sum $5\,b$.

8. The sides of a triangle form an A. P. with a common difference 2. Find the cosine of the largest angle, if the longest side is twice the shortest.

9. Find the angles of a triangle if they form an A. P. with $d = 5°$.

10. Between every pair of consecutive terms of the G. P. 1, 2, 4, 8, . . . insert a new term so that the result is again a G. P.

11. As in exercise 10 for the G. P. $a, ar, ar^2, \ldots$.

12. In a G. P. of 10 terms, the sum of the even terms is 30 and of the odd terms 60. Find a and r.

13. Find four numbers in G. P. such that the product of the first and last is 400 and the quotient of the middle two is 14.

14. Find three numbers in G. P. such that their sum is h, the sum of their squares k.

15. If a tree, now 4 inches in diameter, increases its diameter 5% each year, how thick will it be in 20 years ?

16. A seed yields a plant from which 4 new seeds are obtained. How many seeds are available from the 10th generation of plants ?

17. An Indian potentate offered to reward the inventor of the game of chess as follows: one grain of wheat for the first square on the chessboard, 2 for the second, 4 for the third, and so on, doubling each time for the 64 squares. What would be the cash value of this reward, with wheat at $1.00 a bushel, allowing a million grains to the bushel?

18. A right triangle has a hypotenuse 2 ft., angle 30°. From the vertex of the right angle a ⊥ is dropped on the hypotenuse, forming a new right triangle which is treated similarly, and so on indefinitely. Find the sum of all the ⊥s so obtainable.

19. The altitude of an equilateral triangle is a. A circle is inscribed in it, and in this circle a new equilateral triangle. The operation is repeated on the new triangle, and so on indefinitely. Find the sum of the altitudes and of the perimeters of all triangles so obtainable.

20. Find the sum of the perimeters and of the areas of all the circles in exercise 19.

Interest and Annuities. — This subject affords a simple and useful application of the theory of progressions.

192. Interest. — Let P denote a sum of money loaned, or *principal*, and r the yearly rate of interest expressed in fractions of a dollar. Then the amount of P dollars in one year is

$$A_1 = P(1 + r).$$

If principal plus interest for one year is allowed to run a second year, the amount at the end of the second year is

$$A_2 = A_1(1 + r) = P(1 + r)^2,$$

and so on.

Hence if A_n be the amount of P dollars in n years, interest at rate r compounded annually, we have

$$(1) \qquad \boldsymbol{A_n = P(1 + r)^n}.$$

If interest is compounded every t years instead of annually, then, after n compoundings, the amount is

$$(1') \qquad \boldsymbol{A_n = P(1 + rt)^n}.$$

Thus if we want the amount of $100 at the end of 2 years, interest 4 per cent compounded quarterly, we have,

$$P = \$100;\ r = \tfrac{4}{100};\ t = \tfrac{1}{4};\ n = 8.$$

Then $A_n = 100(1 + .04 \times \frac{1}{4})^8 = \$100(1.01)^8 = \$108.25.$

193. Annuities. — *An annuity is a sum of money payable yearly, or at other stated periods.*

Let A be the amount of each payment, r the yearly rate of interest, n the number of payments to be made.

Assuming the first payment now due, and that each payment is put at interest, compounded annually, what is the total amount accrued when the last payment has been made?

The first payment is at interest $n - 1$ years, its amount $A(1 + r)^{n-1}$; the second $n - 2$ years, its amount $A(1 + r)^{n-2}$; and so on, to the payment next before the last, which is at interest one year, its amount $A(1 + r)$; the last payment amounts to A. The total amount S is therefore

$$S = A + A(1 + r) + A(1 + r)^2 + \cdots + A(1 + r)^{n-1}, \text{ or}$$

$$(2) \qquad S = A\frac{(1 + r)^n - 1}{1 + r - 1} = A\frac{(1 + r)^n - 1}{r}.$$

Present Worth. — How much cash in hand, placed at interest compounded annually, will amount to the sum S just obtained when the last payment is made, that is, in $n - 1$ years?

Let Q be the amount required, called the *present worth* of the annuity.

Let Q_1 be the sum which with interest will yield in $n - 1$ years the amount of the first payment, or $A(1 + r)^{n-1}$. Then

$$Q_1(1 + r)^{n-1} = A(1 + r)^{n-1} \quad \text{or} \quad Q_1 = A.$$

Let Q_2 be the sum which with interest for $n - 1$ years will yield the amount of the second payment, or $A(1 + r)^{n-2}$. Then

$$Q_2(1 + r)^{n-1} = A(1 + r)^{n-2} \quad \text{or} \quad Q_2 = \frac{A}{1 + r}.$$

Similarly if $Q_3, Q_4, \ldots Q_n$ be the present worths of the 3rd, 4th, ... last payments of the annuity we have,

$$Q_3 = \frac{A}{(1 + r)^2}, \quad Q_4 = \frac{A}{(1 + r)^3}, \quad \ldots, \quad Q_n = \frac{A}{(1 + r)^{n-1}}.$$

Hence

$$Q = Q_1 + Q_2 + \cdots + Q_n = A\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^{n-1}}\right).$$

The sum in the parentheses is a G. P. with ratio $\frac{1}{1+r}\cdot$ Applying the formula and reducing,

$$(3) \qquad Q = A\frac{(1+r)^n - 1}{r(1+r)^{n-1}}.$$

194. Exercises.

1. Find the amount of \$1412 in 19 years at 4%, interest compounded annually.

2. Find the present worth of an annuity of \$100, there being 20 annual payments of which the first is now due.

3. Find the amount of \$1000 in 10 years at 4%, interest compounded quarterly.

4. Find the amount of \$1000 in 20 years at 4%, interest compounded semi-annually.

5. In how many years will a sum of money double itself at 5% simple interest ?

6. In how many years will a sum of money double itself at 5%, interest compounded annually ?

7. An annuity of \$100 is to begin in 10 years from date and to run 10 years. Find its present worth if money brings 5% compound interest.

8. Find the present worth of a perpetual annuity of A dollars, compound interest r%, the first payment now due. ($Q = Q_1 + Q_2 + Q_3 + \cdots$ ad inf.).

9. As in exercise 8, except that the first payment falls due in m years.

CHAPTER XII

Infinite Series

195. Limit of a Variable Quantity. — *When a variable quantity changes in such a way that it approaches a fixed numerical value, so that the difference between the variable and the fixed quantity becomes and remains less than any assignable magnitude, however small, then the fixed quantity is called the limit of the variable.*

For example, as x varies the variable quantity $1 + x$ can be made to differ from 1 by less than any small quantity e, by simply taking $|x| < e$, and the nearer x is to 0, the nearer will $1 + x$ be to 1. Hence, *as x approaches 0, the limit of $1 + x$ is 1.* As an equation this is expressed by

$$\lim_{x \doteq 0} (1 + x) = 1. \quad (\doteq \text{ is read "approaches."})$$

Exercise. Show that:

$$(a) \lim_{x \doteq 0} \frac{1}{1+x} = 1; \quad (b) \lim_{x \doteq 1} \left(1 + \frac{1}{x}\right) = 2; \quad (c) \lim_{x \doteq 0} \log (1 + x) = 0;$$

$$(d) \lim_{n \doteq 10} \left(1 - \frac{10}{n}\right) = 0; \quad (e) \lim_{x \doteq 0} e^x = 1; \quad (f) \lim_{n \doteq \infty} \left(1 + \frac{1}{n}\right)^{\frac{1}{n}} = 1.$$

196. Infinite Series. — *A sequence or succession of terms, u_1, u_2, u_3, . . . , u_n, . . . , unlimited in number, is called an infinite series.*

The sum of the first n terms of a sequence we denote by S_n. Then

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n.$$

As n increases and we form the sum of more and more terms of the sequence, one of three alternatives is open to S_n, namely:

(a) S_n approaches a fixed limit S, which is then called the sum of the infinite series, and the series is said to *converge*.

(b) S_n increases without limit; the infinite series then has no sum and is said to *diverge*.

(c) S_n oscillates; the infinite series has no sum but oscillates, and is again said to *diverge*.

Examples.

(a) $$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\cdots.$$

$$S_n=\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}=\frac{1}{2}\frac{(\frac{1}{2})^n-1}{\frac{1}{2}-1}=1-\left(\frac{1}{2}\right)^n. \quad \text{(188.)}$$

$\therefore \quad \lim\limits_{n\doteq\infty} S_n=1=S.$ The series converges to the value 1,

or, $$\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}+\cdots=1. \quad [(188), \text{figure.}]$$

(b) $$1+2+3+\cdots+n+\cdots.$$

$S_n=1+2+3+\cdots+n$; then obviously S_n increases without limit as more and more terms are added. Hence the given series has no sum, and diverges.

(c) $$1-1+1-1+\cdots.$$

Here $S_1=1$; $S_2=1-1=0$; $S_3=1-1+1=1$; $S_4=0$; and so on indefinitely. S_n oscillates from 0 to 1 as n varies, the series is oscillatory and has no sum. We say that it diverges.

197. *To show that an infinite series converges, it must be shown that S_n, the sum of its first n terms, approaches a definite limit as n increases indefinitely. When such limit does not exist, the series is divergent.*

The direct method of determining whether a given series converges or diverges is to form the sum of its first n terms S_n, and let n increase indefinitely. This method is applicable only in the few cases where a formula for S_n is available. The standard case is that of the infinite geometric progression,

$$a+ar+ar^2+\cdots+ar^{n-1}+\cdots$$

Here $$S_n=a+ar+ar^2+\cdots+ar^{n-1}=a\frac{1-r^n}{1-r}.$$

When r is numerically less than 1, i.e., $|r|<1$, then r^n approaches 0 as n increases and

$$\lim_{n=\infty} S_n=a\frac{1}{1-r}=S.$$

When $r=1$,

$$S_n=a+a+\cdots+a=na.$$

Hence S_n increases without limit when n increases. When $|r| > 1$, r^n increases indefinitely with n; hence S_n does the same. Therefore, *the geometric series,* $a + ar + ar^2 + \cdots$, *converges when* $|r| < 1$, *and diverges when* $|r| \geqq 1$.

Putting $a = 1$, we see that the *simple power series,* $1 + x + x^2 + \cdots$, *converges when* $|x| < 1$ *and diverges when* $|x| \geqq 1$.

198. We next consider indirect methods for establishing the convergence or divergence of a given infinite series.

Theorem 1. *When an infinite series converges, its nth term approaches zero as a limit when n increases.*

Proof. Let the convergent series be $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$. Then $S_n = u_1 + u_2 + \cdots + u_n$ and $S_{n-1} = u_1 + u_2 + \cdots + u_{n-1}$.

Hence $$u_n = S_n - S_{n-1}.$$

By taking n large enough, both S_n and S_{n-1} can be made to differ from the sum of the series and hence from each other by as little as we please; hence their difference, u_n, can be made to differ from zero by less than any assignable small quantity.

$$\therefore \quad \lim_{n \doteq \infty} u_n = 0.$$

This is a necessary condition for the convergence of any series.

Test for Divergence. — From *Theorem* 1 we infer that *an infinite series diverges whenever* $\lim\limits_{n \doteq \infty} u_n \neq 0$.

199. Alternating Series. — A series whose terms are alternately $+$ and $-$ is called an *alternating series.*

Theorem 2. *An alternating series converges provided that* (*a*) *each term is numerically less than the preceding, and* (*b*) *the limit of the nth term is zero as n increases indefinitely.*

Proof. Let the series be

$$u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots.$$

Write this in the two forms,

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \cdots;$$
$$u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots.$$

Each set of parentheses incloses a positive quantity according to condition (*a*) of the theorem; hence assuming that $u_1, u_2, u_3, \ldots$ are themselves positive quantities, the first form shows that the sum of the series is positive, i.e., > 0, and the second that the sum is less than the first term u_1. Also, since $\lim_{n \doteq \infty} u_n = 0$, the sum cannot oscillate. Hence the series converges to a value between 0 and its first term.

Example. The alternating series,

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

converges to a value between 0 and 1.

200. Absolute Convergence. — *A series is said to converge absolutely when it remains convergent if all its terms are taken positively.*

Thus if $u_1, u_2, u_3, \ldots$ be in part negative and in part positive, the series

$$u_1 + u_2 + u_3 + \cdots$$

converges absolutely provided that the series

$$|u_1| + |u_2| + |u_3| + \cdots$$

converges.

Exercise. Show that the series

$$1 + x + x^2 + \cdots \quad \text{and} \quad a + ax + ax^2 + \cdots$$

both converge absolutely when $|x| < 1$.

201. The Comparison Test.

Let $$u_1 + u_2 + u_3 + \cdots$$

be a series known to converge absolutely or to diverge.

Let $$v_1 + v_2 + v_3 + \cdots$$

be a series to be tested for convergence or divergence. Then,

(a) *If the u-series converges absolutely and, for all values of n, v_n is numerically less than u_n, the v-series also converges absolutely;*

(b) *If the u-series diverges and v_n is numerically greater than u_n, and if all the terms of the v-series have the same sign, the v-series also diverges.*

Proof.

Let $$U_n = |u_1| + |u_2| + |u_3| + \cdots + |u_n|$$

and $$V_n = |v_1| + |v_2| + |v_3| + \cdots + |v_n|.$$

Then by condition (a), U_n approaches a limit, say U, as $n \doteq \infty$, and also, $V_n < U_n$. Hence, since V_n must increase steadily with

n, but is always less than U_n, it must approach a limit V, less than U. Hence the v-series converges.

Under condition (b), U_n increases without limit, and also, $V_n > U_n$. Hence V_n also increases without limit and the v-series diverges.

Standard Test Series. (For use in Comparison Test.)

(1) $a + ax + ax^2 + \cdots + ax^n + \cdots$, ⎫ Conv. when $|x| < 1$;
(2) $1 + x + x^2 + \cdots + x^n + \cdots$, ⎭ Div. when $|x| \geqq 1$.

(3) $1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots$, Convergent.

(4) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$, Divergent.

(5) $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$, ⎫ Conv. when $p > 1$; ⎭ Div. when $p \leqq 1$.

The first three of these series are geometric progressions and have already been considered.

Series (4) can be shown to diverge by grouping its terms thus:

$$1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + (\tfrac{1}{9} + \tfrac{1}{10} + \cdots + \tfrac{1}{16}) + \cdots.$$

We can form in this way an infinite number of parentheses, each of which is $> \frac{1}{2}$. Hence the sum is infinite.

Series (5) is, term for term, greater than or equal to (4), when $p \leqq 1$; hence for these values of p the series diverges, by condition (b) above. When $p > 1$, the series is shown to converge by grouping its terms as follows:

$$\frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \cdots + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right) + \cdots.$$

Considering each group of terms as a single quantity, we see that this series is less, term for term, than the series

$$1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots,$$

or

$$1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots.$$

But this is a G. P. with ratio $\frac{1}{2^{p-1}}$, and hence converges. Therefore the given series converges.

Examples.

1. The series $1+\frac{1}{2^2}+\frac{1}{3^3}+\cdots+\frac{1}{n^n}+\cdots$ converges; for it is less, term for term, than (3).

2. The series $1+\frac{1}{\log_{10}2}+\frac{1}{\log_{10}3}+\cdots+\frac{1}{\log_{10}n}+\cdots$ diverges; for it is greater, term for term, than (4).

202. The Ratio Test. — *The series* $u_1+u_2+u_3+\cdots+u_n+\cdots$ *converges absolutely if, beginning at some point in the series, the ratio* $u_n \div u_{n-1}$ *becomes and remains numerically less than a fixed positive number which is itself less than* 1.

Proof. Assume that

$$\left|\frac{u_n}{u_{n-1}}\right| < r < 1 \text{ for all values of } n > N,$$

N being a fixed positive integer.

Then $$|u_n| < r|u_{n-1}| \text{ when } n > N.$$

Hence putting $n = N+1, N+2, \ldots$, we have

$$|u_{N+1}| < r|u_N|;$$
$$|u_{N+2}| < r|u_{N+1}| < r^2|u_N|;$$
$$|u_{N+3}| < r|u_{N+2}| < r^3|u_N|;$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Adding, we have

$$|u_{N+1}|+|u_{N+2}|+u_{N+3}|+\cdots < |u_N|(r+r^2+r^3+\cdots).$$

Writing the given series in two parts,

$$(u_1+u_2+\cdots+u_N)+(u_{N+1}+u_{N+2}+u_{N+3}+\cdots),$$

we see that the first part, formed of N terms where N is a fixed finite integer, must have a finite sum. The second part cannot exceed the left member of the last inequality above, hence is less than the right member of that inequality. But the series $r+r^2+r^3+\cdots$ converges and has a finite sum, since it is a G. P. with ratio $r<1$. Hence the sum in the second pair of parentheses has a finite limit, and the given series converges.

Similarly it can be shown that the series diverges when the test-ratio $u_n \div u_{n-1}$ becomes and remains greater than 1, or even when it approaches 1 from the upper side.

When the test-ratio $u_n \div u_{n-1}$ is at first less than 1, but approaches 1 as n increases, this method gives no information about the series.

Examples.

1. $1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} + \cdots .$

Here $\left|\frac{u_n}{u_{n-1}}\right| = \frac{1}{n}$, which approaches 0 as $n \doteq \infty$. Hence the ratio test is satisfied and the series converges.

2. $\sin x + 2 \sin^2 x + 3 \sin^3 x + \cdots + n \sin^n x + \cdots$

$$\left|\frac{u_n}{u_{n-1}}\right| = \left|\frac{n \sin^n x}{(n-1)\sin^{n-1} x}\right| = \frac{n}{n-1}\,|\sin x|.$$

As $n \doteq \infty$, $\frac{n}{n-1} \doteq 1$, and if we choose x different from an odd multiple of $\frac{\pi}{2}$, so that $|\sin x| < 1$, we can take n so large that the test-ratio will be less than r, where r is less than 1. We need only take $x < \sin^{-1} r \frac{n-1}{n}$.

Hence the series converges for any value of x which is not a multiple of $\frac{\pi}{2}$.

3. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots .$

$\left|\frac{u_n}{u_{n-1}}\right| = \frac{n-1}{n}$, which approaches 1 from the lower side. Hence the test fails.

4. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots .$

$$\left|\frac{u_n}{u_{n-1}}\right| = \frac{n}{n+1} \div \frac{n-1}{n} = \frac{n^2}{n^2-1}.$$

Here the test-ratio is greater than 1, approaching 1 from the upper side as $n \doteq \infty$. Hence the series diverges. This series may also be shown to diverge by comparison with (4) of (**201**).

203. Exercises. Test the following series:

1. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots .$

2. $1 + \frac{x}{2} + \frac{x^2}{3} + \cdots + \frac{x^n}{n+1} + \cdots .$

3. $1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots .$

4. $\cos x + \cos^2 x + \cdots + \cos^n x + \cdots .$

5. $\tan x + \tan^2 x + \cdots + \tan^n x + \cdots .$

6. $\sin^{-1} x + (\sin^{-1} x)^2 + \cdots + (\sin^{-1} x)^n + \cdots .$

7. $\log_{10} x + (\log_{10} x)^2 + \cdots + (\log_{10} x)^n + \cdots .$

8. $\frac{1 \cdot 2}{3 \cdot 4} + \frac{2 \cdot 2}{4 \cdot 5} + \frac{3 \cdot 4}{5 \cdot 6} + \cdots .$

9. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots.$

10.* $\underline{|1}\, x + \underline{|2}\, x^2 + \underline{|3}\, x^3 + \cdots + \underline{|n}\, x^n + \cdots.$*

11. $1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots.$

12. $x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots.$

13. $1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \cdots.$

14. $1 - \frac{1}{2} x + \frac{1 \cdot 3}{2 \cdot 4} x - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^2 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} x^3 - \cdots.$

15. $x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \cdots.$

* $\underline{|n} \equiv 1 \cdot 2 \cdot 3 \cdots n.$

CHAPTER XIII

FUNCTIONS. DERIVATIVES. MACLAURIN'S SERIES

204. Functions. — Let x denote a variable quantity and y a quantity whose value depends on that of x. Then y is said to be a *function of* x. Thus

$$y = x^2 + 1, \qquad y = a^x, \qquad y = \sin(ax + b)$$

are all functions of x.

As an equation, we indicate that y is a function of x by writing

$$y = f(x).$$

When a body is dropped from rest, the space s (ft.) fallen through in the time t (seconds) is $s = \frac{1}{2}gt^2$. Here s is a function of t, or

$$s = f(t); \qquad f(t) = \tfrac{1}{2}gt^2.$$

When a train is running at 30 miles an hour, the space s (miles) covered in the time t (hours) is $s = 30\,t$. Hence

$$s = f(t); \qquad f(t) = 30\,t.$$

When the relation between y and x is given by an equation of the form $y = f(x)$, y is called an *explicit function* of x.

Suppose the relation between x and y to be given in the form,

$$x^2 + y^2 = 1.$$

Here y is not given directly in terms of x, but nevertheless the value of y depends on that of x; for when we substitute for x first one value and then another we get in general different values of y on solving the equation. In such case y is called an *implicit function* of x.

As other examples, we have

$$y^2 = 4\,x; \qquad \sin(x + y) = 1; \qquad a^x + a^y = b.$$

205. Variation of Functions. — Consider the relation $y = x^2$. When $x = a$, then $y = a^2$; when $x = a + h$, $y = (a + h)^2$.

As x changes from a to $a+h$, y changes from a^2 to $(a+h)^2$. The total change in x is h, and the corresponding change in y is $(a+h)^2 - a^2$ or $2\,ah + h^2$.

Let us designate a change in x by Δx (read "increment of x," or "delta x") so that in this example $\Delta x = h$; let the corresponding change in y be Δy, so that we have in this case

$$\Delta y = 2\,ah + h^2 = 2\,a\,\Delta x + \overline{\Delta x}^2.$$

In general, if $y = f(x)$, then to the values x and $x + \Delta x$ of the variable x correspond the values $f(x)$ and $f(x+\Delta x)$* of y. Hence the change in y, corresponding to the change Δx in x, is

$$\Delta y = f(x+\Delta x) - f(x).$$

Continuous Function. — When $\Delta y \doteq 0$ with Δx, y is called a continuous function of x. We assume all our functions to be continuous unless the contrary is stated.

Exercises.

1. Given $y = x^2$. Calculate Δy, when $x = 2$ and $\Delta x = 0.1$.
2. As in exercise 1, when $y = \sqrt{x}$.
3. As in exercise 1, when $y = x^3$.
4. As in exercise 1, when $y = 10^x$.
5. Given $y = \sin x$. Calculate Δy, when $x = 45°$ and $\Delta x = 5°$.
6. As in 5, when $x = 30°$ and $\Delta x = 1°$.
7. As in 5, when $x = 1$ and $\Delta x = 0.01$.

206. Difference Quotient. — The fraction

$$\frac{\text{change in } y}{\text{change in } x}, \quad \text{or} \quad \frac{\Delta y}{\Delta x},$$

is called the *difference quotient of y relative to x.*

Thus, if $y = x^2$, then $\Delta y = (x+\Delta x)^2 - x^2 = 2\,x\,\Delta x + \overline{\Delta x}^2$.

Hence the difference quotient is

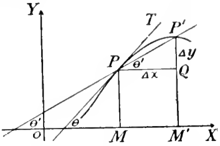
$$\frac{\Delta y}{\Delta x} = \frac{2\,x\,\Delta x + \overline{\Delta x}^2}{\Delta x} = 2\,x + \Delta x.$$

We shall abbreviate Difference Quotient by writing D. Q.

Exercises. Calculate the D. Q. in the exercises of **(205)**.

* $f(x+\Delta x)$ stands for the result obtained by replacing x by $x+\Delta x$ in $f(x)$.

207. The D. Q., $\frac{\Delta y}{\Delta x}$, geometrically. — Let the curve in the figure represent a part of the graph of the equation $y = f(x)$.



Let P be a point on the curve having coördinates $(x = OM,\ y = MP)$, and P' a second point $(x + \Delta x = OM',\ y + \Delta y = M'P')$.

Let the secant PP' make an angle θ' with the x-axis.

Draw $PQ \parallel OX$. Then from $\triangle PQP'$,

$$\tan \theta' = \frac{\Delta y}{\Delta x}.$$

Slope. — The *tangent of the angle* which a line makes with the x-axis is called the *slope* of the line.

Hence, *the difference quotient, $\frac{\Delta y}{\Delta x}$, is the slope of the secant drawn through the points (x, y) and $(x + \Delta x, y + \Delta y)$.*

208. Limit of D. Q. = Slope of Tangent. — Let the point P' move back along the curve and approach the point P. Then Δx, and in general also Δy, approach 0.

Suppose now that as Δx approaches 0 the D. Q. approaches a definite limit, m.

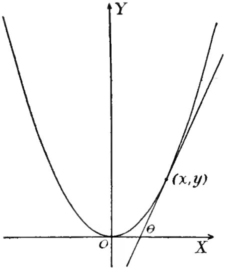
Then the line through the point (x, y) having the slope m is called the *tangent to the curve* $y = f(x)$, (x, y) being the point of contact.

In the figure, as P' approaches P, the secant line PP' gradually rotates about P and approaches a limiting position PT, which is *defined* to be the tangent to the curve at P.

If θ be the angle which the tangent to the curve at $P = (x, y)$ makes with the x-axis, then

$$\tan \theta = \lim_{\Delta x \doteq 0}\left(\frac{\Delta y}{\Delta x}\right) \quad \left\{\text{read, "tangent of } \theta \text{ equals the limit of } \frac{\Delta y}{\Delta x} \text{ as } \Delta x \text{ approaches 0."}\right.$$

When $\frac{\Delta y}{\Delta x}$ approaches a definite limit a tangent is thereby determined. When such limit is indeterminate, the tangent does not exist, or several tangents may be drawn at P. We shall consider only cases where a single determinate tangent exists.



$y = x^2$
$\tan\theta = 2x$

209. Examples.

1. $y = x^2$.

$$y + \Delta y = (x + \Delta x)^2$$
$$= x^2 + 2x\,\Delta x + \overline{\Delta x}^2.$$
$$\therefore \quad \Delta y = 2x\,\Delta x + \overline{\Delta x}^2$$

and $$\frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

Hence $$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 2x = \tan\theta.$$

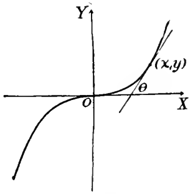
Here the slope of the tangent at any point equals twice the abscissa.

2. $y = \frac{1}{27}x^3$.

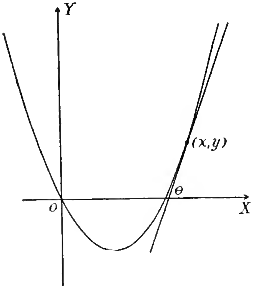
$$y + \Delta y = \tfrac{1}{27}(x + \Delta x)^3$$
$$= \tfrac{1}{27}(x^3 + 3x^2\,\Delta x + 3x\,\overline{\Delta x}^2 + \overline{\Delta x}^3).$$
$$\therefore \quad \Delta y = \tfrac{1}{27}(3x^2\,\Delta x + 3x\,\overline{\Delta x}^2 + \overline{\Delta x}^3)$$

and $$\frac{\Delta y}{\Delta x} = \tfrac{1}{27}(3x^2 + 3x\,\Delta x + \overline{\Delta x}^2).$$

Hence $$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \frac{1}{9}x^2 = \tan\theta.$$



$y = \frac{1}{27}x^3$
$\tan\theta = \frac{1}{9}x^2$



$y = x^2 - 2x$
$\tan\theta = 2x - 2$

3. $y = x^2 - 2x$.

$$y + \Delta y = (x + \Delta x)^2 - 2(x + \Delta x)$$
$$= x^2 + 2x\,\Delta x + \overline{\Delta x}^2 - 2x - 2\,\Delta x$$
$$= x^2 - 2x + (2x - 2)\,\Delta x + \overline{\Delta x}^2.$$
$$\therefore \quad \Delta y = (2x - 2)\,\Delta x + \overline{\Delta x}^2$$

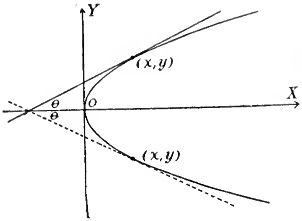
and $$\frac{\Delta y}{\Delta x} = (2x - 2) + \Delta x.$$

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 2x - 2 = \tan\theta.$$

4. $y^2 = x$. Here y is an implicit function of x. Solving, we have

$$y = \pm\sqrt{x}.$$

The upper sign gives that part of the curve lying above the x-axis, the lower sign the part below the axis. We consider first the upper sign only.



$y^2 = x.\quad \tan\theta = \pm\dfrac{1}{2\sqrt{x}}$

Then

$$y = \sqrt{x} \quad\text{and}\quad y + \Delta y = \sqrt{x + \Delta x}.$$

$$\therefore\quad \Delta y = \sqrt{x + \Delta x} - \sqrt{x}.$$

Multiplying and dividing by $\sqrt{x + \Delta x} + \Delta x$, we get

$$\Delta y = \frac{(\sqrt{x+\Delta x} - \sqrt{x})(\sqrt{x+\Delta x} + \sqrt{x})}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \frac{\Delta x}{\sqrt{x + \Delta x} + \sqrt{x}}.$$

Hence $$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}},$$

and $$\lim_{\Delta x \doteq 0}\frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}} = \tan\theta.$$

For the lower part of the curve, replace $\sqrt{x}$ by $-\sqrt{x}$.

5. $x^2 + y^2 = 100.$

Solving for y, we get

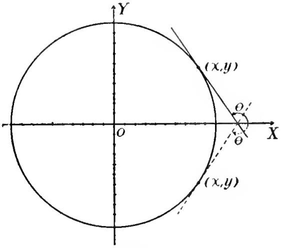
$$y = \pm\sqrt{100 - x^2}.$$

Considering first only the upper half of the circle (figure) we have

$$y = \sqrt{100 - x^2}\,;$$

$$y + \Delta y = \sqrt{100 - (x + \Delta x)^2}.$$

$$\therefore\quad \Delta y = \sqrt{100 - (x + \Delta x)^2} - \sqrt{100 - x^2}.$$



$x^2 + y^2 = 100$

$\tan\theta = \mp\dfrac{x}{\sqrt{100 - x^2}}$

Multiplying and dividing by the sum of the two radicals,

$$\Delta y = \frac{-2x\,\Delta x - \Delta x^2}{\sqrt{100 - (x + \Delta x)^2} + \sqrt{100 - x^2}}.$$

Hence $$\frac{\Delta y}{\Delta x} = -\frac{2x + \Delta x}{\sqrt{100 - (x + \Delta x)^2} + \sqrt{100 - x^2}},$$

and $$\lim_{\Delta x \doteq 0}\frac{\Delta y}{\Delta x} = -\frac{2x}{2\sqrt{100 - x^2}} = -\frac{x}{\sqrt{100 - x^2}} = \tan\theta.$$

At any point on the lower half of the circle, $\tan\theta = +\dfrac{x}{\sqrt{100 - x^2}}$.

In all these examples the slope of the tangent at any given point may be obtained by substituting the abscissa of the point in the value of $\tan\theta$.

Exercises. Calculate the slopes of the tangents at any point (x, y) on the following curves:

1. $y = \frac{1}{8}x^3$.
2. $y = 2x^2 - 3x$.
3. $y = x^3 - x$.
4. $y^2 = 4x$.
5. $y^2 = -9x$.
6. $x^2 + y^2 = 1$.
7. $x^2 - y^2 = 1$.
8. $9x^2 + 16y^2 = 144$.
9. $4x^2 - y^2 = 4$.

Calculate the slope in each of these examples when $x = 1$. Note the results in exercises 6 and 7 and explain.

210. Derivative. — The expression $\lim_{\Delta x \doteq 0}\left(\frac{\Delta y}{\Delta x}\right)$ occurs so frequently in mathematics that a special name is applied to it. Starting with y as any given function of x, say $f(x)$, we can derive from this a second function of x as follows. Calculate $f(x + \Delta x) - f(x)$ or Δy, divide by Δx, and pass to the limit by allowing Δx to approach zero. Call the new function of x so obtained $f'(x)$, so that

$$f'(x) = \lim_{\Delta x \doteq 0}\left(\frac{\Delta y}{\Delta x}\right).$$

This is called the *first derived function of* $f(x)$ or the *first derivative of* $f(x)$, and the expression

$$\lim_{\Delta x \doteq 0}\left(\frac{\Delta y}{\Delta x}\right)$$

is called the *first derivative of* y *with respect to* x. It is usually written in one of the forms

$$\lim_{\Delta x \doteq 0}\left(\frac{\Delta y}{\Delta x}\right) = D_x y, \text{ or } = \frac{dy}{dx}.$$

Hence the slope of the tangent to the curve $y = f(x)$ at a point (x, y) is

$$\tan\theta = D_x y = \frac{dy}{dx}.$$

211. Calculation of Derivatives. — We have already calculated the derivative of y with respect to x in a number of cases. We now obtain a few simple formulas for the calculation of derivatives. Three steps are involved in every case: (1) *the calculation of* Δy, (2) *division by* Δx, (3) *evaluation of the limit as* $\Delta x \doteq 0$. *We shall assume that such a limit exists.*

Formulas for Calculating Derivatives.

I. $D_x(c) = 0$, c being a constant.

(1) For if c is a constant its change is 0, hence $\Delta c = 0$.

(2) Therefore $\frac{\Delta c}{\Delta x} = 0.$

(3) Hence $\lim_{\Delta x \doteq 0}\frac{\Delta c}{\Delta x} = 0$ or $D_x(c) = 0.$

II. $D_x(cy) = c\,D_xy$, c being any constant.

Proof.

(1) The increment in y being Δy, the increment in cy will be $c\,\Delta y$.

(2) Dividing by Δx, the D. Q. of cy relative to x is $c\dfrac{\Delta y}{\Delta x}$.

(3) Let $\Delta x \doteq 0$. Then c does not change, while $\dfrac{\Delta y}{\Delta x}$ becomes $D_x(y)$. Hence

$$D_x(cy) = \lim_{\Delta x \doteq 0} c\frac{\Delta y}{\Delta x} = c\,D_xy.$$

III. When y is a sum of several functions of x, as

$$y = u + v + w + \cdots, \quad \text{where} \quad u, v, w, \ldots$$

are functions of x, then

$$D_xy = D_xu + D_xv + D_xw + \cdots.$$

Proof. When x takes an increment Δx, let the corresponding changes in $u, v, w, \ldots$ be $\Delta u, \Delta v, \Delta w, \ldots$ respectively. The total change in y is, therefore,

(1) $$\Delta y = \Delta u + \Delta v + \Delta w + \cdots.$$

(2) Then $$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \frac{\Delta w}{\Delta x} + \cdots.$$

(3) Let $\Delta x \doteq 0$. Then by definition (**210**), $\dfrac{\Delta y}{\Delta x}$ approaches D_xy, $\dfrac{\Delta u}{\Delta x}$ approaches D_xu, etc. Hence

$D_xy = D_xu + D_xv + D_xw + \cdots$, when $y = u + v + w + \cdots$.

IV. Let y be the product of two continuous functions of x, say u and v.

$$y = u \cdot v.$$

When x is changed to $x + \Delta x$, let u change to $u + \Delta u$ and v to $v + \Delta v$. Then

$$y + \Delta y = (u + \Delta u)(v + \Delta v) = uv + u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v.$$

(1) Hence $$\Delta y = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v.$$

(2) Then $$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$

(3) Let $\Delta x \doteq 0$. Then $\frac{\Delta y}{\Delta x}, \frac{\Delta u}{\Delta x}, \frac{\Delta v}{\Delta x}$ approach $D_x y, D_x u$ and $D_x v$ respectively. Also $\Delta u \doteq 0$, since we assume u to be a continuous function of x (**205**). Hence (2) becomes

$$\boldsymbol{D_x y = u\, D_x v + v\, D_x u}, \text{ when } y = u \cdot v.$$

V. Let $y = \frac{u}{v}$, u and v being continuous functions of x.

Then $$y + \Delta y = \frac{u + \Delta u}{v + \Delta v},$$

(1) and $$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\,\Delta u - u\,\Delta v}{v^2 + v\,\Delta v}.$$

(2) Hence $$\frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v^2 + v\,\Delta v}$$

(3) and $$\boldsymbol{D_x y} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \boldsymbol{\frac{v\,D_x u - u\,D_x v}{v^2}}.$$

VI. Let y be a function of u, where u is a function of x. Thus

$$y = u^2 + 2u;\ u = 2x^2 + 1.$$

When x changes to $x + \Delta x$, u changes to $u + \Delta u$ and y to $y + \Delta y$.

Now $$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

Hence $$\boldsymbol{D_x y = D_u y \cdot D_x u}.$$

Collecting our formulas we have:

(A) $\boldsymbol{D_x c = 0}$.

(B) $\boldsymbol{D_x (cy) = c\, D_x y}$.

(C) $\boldsymbol{D_x (u + v + w + \cdots) = D_x u + D_x v + D_x w + \cdots}$.

(D) $\boldsymbol{D_x (u \cdot v) = u\, D_x v + v\, D_x u}$.

(E) $\boldsymbol{D_x \left(\frac{u}{v}\right) = \frac{v\,D_x u - u\,D_x v}{v^2}}$.

(F) $\boldsymbol{D_x y = D_u y \cdot D_x u}$.

212. We next derive the following **standard formulas**:

(G) $\boldsymbol{y = x^n;\ D_x y = n x^{n-1}}$.

(H) $\boldsymbol{y = \log x;\ D_x y = \frac{1}{x}}$.

(I) $y = a^x$; $D_x y = a^x \log a$.
(J) $y = \sin x$; $D_x y = \cos x$.
(K) $y = \cos x$; $D_x y = -\sin x$.

(G) $y = x^n$; assume n to be a positive integer.

$$y + \Delta y = (x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2} x^{n-2}\overline{\Delta x}^2 + \cdots + \overline{\Delta x}^n.$$

(1) Hence $\Delta y = nx^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2} x^{n-2}\overline{\Delta x}^2 + \cdots + \overline{\Delta x}^n$.

(2) Then $\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{1 \cdot 2} x^{n-2}\Delta x + \cdots + \overline{\Delta x}^{n-1}$.

(3) Let $\Delta x \doteq 0$. All terms on the right of the last equation vanish except the first, and

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = D_x y = nx^{n-1}.$$

The proof when n is not a positive integer will be given after formula **(H)** is derived.

(H) $y = \log x$; $y + \Delta y = \log(x + \Delta x)$.

(1) $\Delta y = \log(x + \Delta x) - \log x = \log \frac{x + \Delta x}{x} = \log\left(1 + \frac{\Delta x}{x}\right)$.

(2) $\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x}\log\left(1 + \frac{\Delta x}{x}\right) = \log\left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}} = \frac{1}{x}\log\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$.

(3) Let $\Delta x \doteq 0$. We must evaluate

$$\lim_{\Delta x \doteq 0} \log\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}.$$

Let $z = \frac{x}{\Delta x}$; then $z \doteq \infty$ when $\Delta x \doteq 0$, provided $x \neq 0$. [$x = 0$ is excluded by our standing assumption of continuity (**205**).] We must now evaluate

$$\lim_{z \doteq \infty} \left(1 + \frac{1}{z}\right)^z.$$

Let $z = 1, 2, 3, \ldots, n$. The corresponding values of $\left(1 + \frac{1}{z}\right)^z$ are 2, 2.25, 2.37, $\ldots$, $\left(1 + \frac{1}{n}\right)^n$. As n increases, these values steadily increase, but always remain less than 3, no matter how large n may be. For, by the Binomial Theorem,

$$\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{n^3} + \cdots$$

to $(n+1)$ terms

$$= 1 + 1 + \frac{\left(1-\frac{1}{n}\right)}{1\cdot 2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\cdot 2\cdot 3} + \cdots \left\{\begin{array}{l}\text{to } (n+1)\\ \text{terms.}\end{array}\right.$$

As n increases, each term of the expansion increases as well as the number of terms. Also all the terms are positive. Hence their sum increases with n. Further compare the above expansion, leaving out the first term $(=1)$, with the geometric progression

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}},$$

whose sum is less than 2. $\left(S = \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}.\right)$

For all values of n, however large, our expansion is less, term for term, than the progression. As $n \doteq \infty$, the sum of the progression approaches 2, hence the expansion, excepting its first term, approaches a limit less than 2. Adding the first term, the limit is less than 3.

This limit is an irrational number denoted by the letter e, and has the approximate value

$$e = 2.7182818 + \cdots.$$

We have now the result that

$$\lim_{z=\infty}\left(1+\frac{1}{z}\right)^z = e$$

when z approaches infinity through positive integral values. The same is true when z increases continuously, but we shall not stop for the proof, which may be found in texts on the calculus.

Then $$\lim_{z=\infty}\log\left(1+\frac{1}{z}\right)^z = \log e,$$

and hence $$D_x(\log x) = \frac{1}{x}\log e.$$

Let us now *take e as the base of our system of logarithms*, so that $\log x$ shall mean $\log_e x$. Then

$$\log e = \log_e e = 1.$$

Hence
$$D_x(\log x) = \frac{1}{x}.$$

Logarithms to the base e are called natural or Naperian logarithms. In the theory of mathematics natural logarithms are in general use, common logarithms, to the base 10, being utilized only for numerical computation.

We can now derive formula (**G**) without any restriction on the value of n.

From
$$y = x^n$$
we have
$$\log y = n \log x. \quad \text{(Base } e.)$$
Hence
$$D_x(\log y) = D_x(n \log x).$$

Now in formula (**F**) replace y by $\log y$ and u by y. It becomes

$$D_x(\log y) = D_y(\log y) \cdot D_x y = \frac{1}{y} D_x(y), \text{ from } (\mathbf{H}).$$

Also
$$D_x(n \log x) = \frac{n}{x}, \text{ from } (\mathbf{B}) \text{ and } (\mathbf{H}).$$

$$\therefore \quad \frac{1}{y} D_x y = \frac{n}{x}$$

or
$$D_x y = \frac{ny}{x}, \text{ where } y = x^n.$$

Hence
$$D_x x^n = \frac{nx^n}{x} = nx^{n-1}.$$

(**I**)
$$y = a^x.$$

Taking logarithms, $\log y = x \log a$.

Hence
$$D_x(\log y) = D_x(x \log a).$$
But
$$D_x(\log y) = \frac{1}{y} D_x y \text{ (see above)}$$
and
$$D_x(x \log a) = \log a.$$
Hence
$$\frac{1}{y} D_x y = \log a,$$
or
$$D_x y = y \log a, \text{ where } y = a^x.$$

Therefore $$D_x a^x = a^x \log a.$$

(J) $$y = \sin x; \qquad y + \Delta y = \sin (x + \Delta x).$$

(1) $$\Delta y = \sin (x + \Delta x) - \sin x = 2 \cos\left(x + \frac{\Delta x}{2}\right) \sin \frac{\Delta x}{2}. \quad \textbf{(158.)}$$

(2) $$\frac{\Delta y}{\Delta x} = \frac{2 \cos\left(x + \frac{\Delta x}{2}\right) \sin \frac{\Delta x}{2}}{\Delta x} = \cos\left(x + \frac{\Delta x}{2}\right) \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}}.$$

(3) Let $\Delta x \doteq 0$. Then

$$\cos\left(x + \frac{\Delta x}{2}\right) \doteq \cos x, \text{ and } \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \doteq 1. \quad \left(\textbf{160.} \text{ Replace } x \text{ by } \frac{\Delta x}{2}.\right)$$

$$\therefore \quad \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \boldsymbol{D_x \sin x = \cos x}.$$

(K) $$y = \cos x; \qquad y + \Delta y = \cos (x + \Delta x).$$

(1) $$\Delta y = \cos (x + \Delta x) - \cos x = -2 \sin\left(x + \frac{\Delta x}{2}\right) \sin \frac{\Delta x}{2}. \quad \textbf{(158.)}$$

(2) $$\frac{\Delta y}{\Delta x} = -\sin\left(x + \frac{\Delta x}{2}\right) \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}}.$$

(3) $$\therefore \quad \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \boldsymbol{D_x \cos x = -\sin x}.$$

By suitable combinations of formulas **(A)** to **(K)** the derivative of any function may be calculated.

213. *Examples.*

1. Calculate $D_x (4 x^3 + 3 x)$.

$$\begin{aligned} D_x (4 x^3 + 3 x) &= D_x (4 x^3) + D_x (3 x) && \text{(C)} \\ &= 4 D_x x^3 + 3 D_x x && \text{(B)} \\ &= 12 x^2 + 3. && \text{(G)} \end{aligned}$$

2. Calculate $D_x\left(\dfrac{e^x}{1 + \log x}\right)$.

$$D_x\left(\frac{e^x}{1+\log x}\right) = \frac{(1+\log x)\,D_x e^x - e^x D_x(1+\log x)}{(1+\log x)^2} \quad \textbf{(E)}$$

$$= \frac{(1+\log x)\,e^x - e^x\dfrac{1}{x}}{(1+\log x)^2} \quad \textbf{(I), (C), (H)},$$

$$= e^x\,\frac{x(1+\log x)-1}{x(1+\log x)^2}.$$

3. Calculate $D_x(3\sin^2 x)$.

$$D_x(3\sin^2 x) = 3\,D_x\sin^2 x \quad \textbf{(B)}$$

$$= 6\sin x\,D_x\sin x \quad \textbf{(F)};\quad (u=\sin x)$$

$$= 6\sin x\cos x.$$

214. Exercises. Calculate $D_x y$ when:

1. $y = 3x^4 + 5x^3$.

2. $y = x^3 + \dfrac{1}{x^3}$.

3. $y = \frac{1}{2}x^{\frac{1}{2}} + \frac{1}{3}x^{\frac{1}{3}}$.

4. $y = x^{-\frac{1}{2}} - 2x^{\frac{1}{2}}$.

5. $y = \dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt[3]{x}}$.

6. $y = \sin x + e^{2x}$.

7. $y = e^{x^2}$.

8. $y = a^{x^3}$.

9. $y = \cos x + \dfrac{1}{\cos x}$.

10. $y = \log(x+2)$.

11. $y = \log(3x^2 - 1)$.

12. $y = e^x \log x$.

13. $y = \sin x \log\cos x$.

14. $y = e^{\sin x}$.

15. $y = \tan x\left(=\dfrac{\sin x}{\cos x}\right)$.

16. $y = \cot x$.

17. $y = \log\tan x$.

18. $y = \sec x$.

215. The Derivative as a Rate of Change. — The difference quotient $\dfrac{\Delta y}{\Delta x}$ gives the *average* rate of change of y relative to x when x changes by an amount Δx. The smaller Δx, the more nearly will the D. Q. represent the actual (or instantaneous) rate of change of y relative to x. Hence the limit of the D. Q. as $\Delta x \doteq 0$ is taken as the *actual rate of change*.

Rule. *To find the rate of change of one quantity relative to another, calculate the derivative of the first quantity with respect to the second.*

Examples.

1. $y = x^2$. Then $D_x y = 2x$.

Hence y changes $2x$ times as fast as x.

2. In the case of a falling body, if s be the space and t the time and the body starts from rest, we have

$$s = \tfrac{1}{2} gt^2.$$

Then $D_t s = gt =$ velocity at time t.

3. Find the rate of change of the volume of a sphere relative to the radius.

$$V = \tfrac{4}{3}\pi r^3; \quad D_r V = 4\pi r^2.$$

That is, the volume of a sphere changes $4\pi r^2$ times as fast as the radius.

216. Exercises. Calculate the rate of change of:

1. y relative to x, when $y = x^3 + x^2$.
2. y relative to x, when $y = \sin x$.
3. y relative to x, when $y = \sin x \cos x$.
4. y relative to x, when $y = \sin^2 x + \cos^2 x$.
5. y relative to x, when $y = e^x$.
6. the volume of a cube relative to its edge.
7. the surface of a cube relative to its edge.
8. the surface of a sphere relative to its radius.
9. the volume of a cylinder relative to its altitude.
10. the volume of a cone relative to the radius of its base.
11. the area of a circle relative to its perimeter.

12. A body starts when $t = 0$ and moves so that the space described in time t (seconds) is $s = 16t^2 + 10$. Find its velocity when $t = 10$; $t = 5$; $t = 0$.

13. The space-time equation being $s = 2t^3 + 3t - 5$, find the velocity at any time t; what is it when $t = 10$; $t = 1$; $t = 0$?

14. As in 13, when $s = 10 \sin\left(3t + \frac{\pi}{4}\right)$.

15. Given two sides and the included angle of a triangle. Calculate the rate of change of the third side relative to each of the given sides and to the given angle.

217. Higher Derivatives. — When y is a function of x, $D_x y$ is in general a new function of x; the derivative of this new function is called the second derivative of y with respect to x and is written $D_x^2 y$. The derivative of the second derivative is called the third derivative, written $D_x^3 y$, and so on.

Examples.

1. $y = x^3$. $\quad D_x y = 3x^2$; $\quad D_x^2 y = 6x$; $\quad D_x^3 y = 6$; $\quad D_x^4 y = 0$.

2. $y = \sin x$. $\quad D_x y = \cos x$; $\quad D_x^2 y = -\sin x$; $\quad D_x^3 y = -\cos x$; etc.

3. $y = x^n$. $\quad D_x y = nx^{n-1}$; $\quad D_x^2 y = n(n-1)x^{n-2}$; . . . .

$$D_x^n y = n(n-1) \ldots 1 = \lfloor n.$$

218. Maclaurin's Series. — Suppose that a given function of x, $f(x)$, can be represented by a converging power series in x, thus:

$$(1) \quad f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots .$$

To find the values of the coefficients $c_0, c_1, c_2 \cdots$. Put $x = 0$ in (1) and we have c_0 determined by

$$f(0) = c_0.$$

To get c_1, calculate $D_x f(x)$ or $f'(x)$ from (1);

$$(2) \quad f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} + \cdots .$$

Put $x = 0$ in (2) and we have c_1 determined by $f'(0) = c_1$. From (2) calculate $D_x f'(x)$ or $f''(x)$;

$$(3) \quad f''(x) = 2c_2 + 2 \cdot 3 c_3 x + \cdots + n(n-1)x^{n-2} + \cdots .$$

Put $x = 0$ in (3) and we have

$$f''(0) = 2c_2 \quad \text{or} \quad c_2 = \tfrac{1}{2} f''(0).$$

Calculating $D_x f''(x)$, or $f'''(x)$, we have

$$(4) \quad f'''(x) = 2 \cdot 3 c_3 + \cdots + n(n-1)(n-2)x^{n-3} + \cdots .$$

When $x = 0$, $\qquad f'''(0) = 2 \cdot 3 c_3; \quad c_3 = \dfrac{1}{2 \cdot 3} f'''(0).$

Similarly,

$$c_4 = \frac{1}{2 \cdot 3 \cdot 4} f^{\text{IV}}(0) = \frac{1}{\lfloor 4} f^{\text{IV}}(0),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$c_n = \frac{1}{n(n-1)\ldots 1} f^{(n)}(0) = \frac{1}{\lfloor n} f^{(n)}(0).$$

Hence

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots + \frac{x^n}{\lfloor n} f^{(n)}(0) + \cdots .$$

Here $f^{(n)}(0)$ is found by differentiating $f(x)$ n times in succession and putting $x = 0$ in the result.

The above result is called *Maclaurin's series* for the function $f(x)$. In obtaining it we have tacitly assumed that, if $f(x)$ be represented by a power series, the derivative $f'(x)$ can be calculated by differentiating the series term by term.

219. *Examples.*

1. Develop e^x in a power series in x.

$$f(x) = e^x;\quad f'(x) = e^x;\quad f''(x) = e^x;\ \ldots;\ f^{(n)}(x) = e^x.$$

Putting $x = 0$, we have

$$f(0) = 1;\quad f'(0) = 1;\quad f''(0) = 1;\ \ldots;\ f^{(n)}(0) = 1.$$

Hence

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots + \frac{x^n}{\lfloor n} + \cdots.$$

This series converges for all values of x, and is used for calculating the value of e^x to any desired degree of approximation.

When $x = 1$,

$$e = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots + \frac{1}{\lfloor n} + \cdots,$$

from which e can be found approximately by taking a few terms of the series.

2. Develop $\sin x$ in a power series in x.

$$f(x) = \sin x;\quad f'(x) = \cos x;\quad f''(x) = -\sin x;\quad f'''(x) = -\cos x, \ldots.$$

When $x = 0$,

$$f(0) = 0;\quad f'(0) = 1;\quad f''(0) = 0;\quad f'''(0) = -1, \text{ etc.}$$

Hence

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots.$$

This series converges for every value of x, and may be used for finding $\sin x$ to any degree of approximation. Thus, put

$$x = 10° = \frac{\pi}{18} \text{ radians.}$$

Then

$$\sin 10° = \frac{\pi}{18} - \frac{1}{6}\left(\frac{\pi}{18}\right)^3 + \frac{1}{120}\left(\frac{\pi}{18}\right)^5 - \cdots.$$

Note. In computing with an alternating series (signs alternately + and −), the error committed in using only a few of the first terms of the series is always numerically less than the first term neglected.

Thus the error in sin 10° as obtained from the three terms written above is less than

$$\frac{1}{5040}\left(\frac{\pi}{18}\right)^7, \text{ or less than } .000\,000\,000\,98.$$

Hence the error is less than 1 unit in the ninth decimal place.

Exercise. Show that

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots.$$

Calculate cos 10° to five places.

3. Develop $\log(1+x)$ in powers of x.

$$f(x) = \log(1+x); \qquad f(0) = \log 1 = 0.$$

$$f'(x) = \frac{1}{1+x}; \qquad f'(0) = 1.$$

$$f''(x) = -\frac{1}{(1+x)^2}; \qquad f''(0) = -1.$$

$$f'''(x) = \frac{2}{(1+x)^3}; \qquad f'''(0) = 2.$$

$$f^{\text{IV}}(x) = \frac{-2\cdot 3}{(1+x)^4}; \qquad f^{\text{IV}}(0) = -2\cdot 3.$$

. . . .

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

This series converges only when $-1 < x \leqq 1$, and hence can be used only when x lies between -1 and $+1$ and for $x = +1$.

Since the base of the logarithm system in $\log(1+x)$ is understood to be e, the last series enables us to calculate the natural or Naperian logarithms of numbers from 0 to 2, exclusive of 0. For $1+x$ ranges from 0 to 2 when x ranges from -1 to $+1$. In particular, when $x = 1$ we have

$$\log_e 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots .$$

This is a convergent alternating series. Since in such a series the error committed by neglecting all terms after a given one is less than that term (**199**)*, 1000 terms of the series would be required to give log 2 correct to three decimal places. The series therefore converges too slowly for practical use. A more serviceable series will be considered in the next chapter.

220. The Binomial Theorem. — When n is a positive integer, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \cdots + x^n.$$

We shall now derive the formula for expanding $(1+x)^n$ in powers of x for any value of n, positive or negative, integral or non-integral.

Let $$f(x) = (1+x)^n.$$

* Apply (**199**) to the neglected part of the given series.

Then

$$f'(x) = n(1+x)^{n-1}; \qquad f'(0) = n.$$
$$f''(x) = n(n-1)(1+x)^{n-2}; \qquad f''(0) = n(n-1)$$
$$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}; \quad f'''(0) = n(n-1)(n-2).$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f^{(m)}(x) = n(n-1)(n-2)\ \ldots\ (n-m+1)\,x^{n-m};$$
$$f^{(m)}(0) = n(n-1)(n-2)\ \ldots\ (n-m+1).$$

Hence by Maclaurin's series,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$
$$+ \frac{n(n-1)\ \cdots\ (n-m+1)}{1 \cdot 2 \cdot\ \ldots\ m}x^m + \cdots,$$

provided that the series on the right, called the *Binomial Series,* converges.

Convergence of the Binomial Series. — Denote the mth term of the series by u_m, the $(m+1)$th term by u_{m+1}. Then

$$u_m = \frac{n(n-1)(n-2)\ \ldots\ (n-m+2)}{1 \cdot 2 \cdot 3 \cdot\ \ldots\ (m-1)}x^{m-1};$$
$$u_{m+1} = \frac{n(n-1)(n-2)\ \ldots\ (n-m+2)(n-m+1)}{1 \cdot 2 \cdot 3 \cdot\ \ldots\ (m-1) \cdot m}x^m.$$

Applying the ratio-test (**202**), we have

$$\frac{u_{m+1}}{u_m} = \frac{n-m+1}{m}x = \left(\frac{n+1}{m} - 1\right)x.$$

The quantity in the last parenthesis is *numerically* less than 1, when m is larger than $n+1$; to secure this we simply start far enough out in the series to make $m > n+1$. Then the ratio $u_{m+1} \div u_m$ will be *numerically* less than x, and hence, *if x be numerically less than* 1, *the series converges. When x is numerically greater than* 1, *the series diverges.* For the ratio $u_{m+1} \div u_m$ equals the product of two factors, $\left(\frac{n+1}{m} - 1\right)$ and x. As m increases the first factor approaches -1 as a limit. Hence if $|x| > 1$, the product will also ultimately be greater than 1 numerically. Finally, when $x = \pm 1$ our binomial reduces to 2^n or 0 respectively and we need not consider the series at all.

We therefore use the binomial series for $(1+x)^n$ only when $|x|<1$.

221. Binomial Series for $(a+b)^n$. — We have

$$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n$$

$$=a^n\left(1+n\frac{b}{a}+\frac{n(n-1)}{1\cdot 2}\frac{b^2}{a^2}+\cdots+\frac{n(n-1)\cdots(n-m+1)}{1\cdot 2\cdot \ldots m}\frac{b^m}{a^m}+\cdots\right)$$

or,

$$(a+b)^n = a^n + na^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2+\cdots$$

$$+\frac{n(n-1)\cdots(n-m+1)}{1\cdot 2\cdot \ldots m}a^{n-m}b^m+\cdots.$$

The series converges when $\left|\frac{b}{a}\right|<1$, that is, when b *is numerically less than* a.

The mth term of the expansion is

$$u_m = \frac{n(n-1)\cdots(n-m+2)}{1\cdot 2\cdot \ldots (m-1)}a^{n-m-1}b^{m-1}.$$

Examples.

1. $\sqrt{1-x}=(1-x)^{\frac{1}{2}}=1-\frac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}x^2-\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}x^3+\cdots$

$$= 1-\tfrac{1}{2}x-\tfrac{1}{8}x^2-\tfrac{3}{48}x^3+\cdots.$$

2. Find an approximate value of $\sqrt{.98}$.

$$\sqrt{.98}=\sqrt{1-.02}=1-\tfrac{1}{2}(.02)-\tfrac{1}{8}(.02)^2-\cdots=.990+.$$

The neglected part of the series is less, term for term, than the G. P.,

$$(.02)^2+(.02)^3+\cdots+(.02)^n+\cdots,$$

whose sum is

$$S=\frac{(.02)^2}{1-.02}=.0004 \text{ approx.}$$

3. Find the 7th term of the expansion of $\sqrt[3]{(2-3\sqrt{x})^4}$ in powers of x.

$$\sqrt[3]{(2-3\sqrt{x})^4}=(2-3\sqrt{x})^{\frac{4}{3}}.$$

Hence $\quad a=2,\ b=-3\sqrt{x},\ n=\frac{4}{3},\ m=7.$

Then $\quad u_7=\dfrac{\frac{4}{3}(\frac{4}{3}-1)(\frac{4}{3}-2)\cdots(\frac{4}{3}-5)}{1\cdot 2\cdot 3\cdot \ldots 6}2^{\frac{4}{3}-6}(-3\sqrt{x})^6=\dfrac{11\sqrt[3]{2}}{72}x^3.$

In this case the expansion converges if

$$|3\sqrt{x}|<2, \quad\text{or}\quad |9x|<4, \quad\text{or}\quad |x|<\tfrac{4}{9}.$$

For negative values of x the expansion would involve imaginary terms because of the presence of $\sqrt{x}$.

222. Exercises. — Write the first four terms of the developments in series of the following functions, and give the values of x for which the series converge.

1. $\tan x$.
2. $\sec x$.
3. $\sin^2 x$.
4. $\sin x^2$.
5. e^{2x}.
6. e^{-x}.
7. $e^x + e^{-x}$.
8. $e^{\frac{x}{a}}$.
9. $e^{-\frac{x}{a}}$.
10. $e^{\frac{x}{a}} + e^{-\frac{x}{a}}$.
11. $\sin x + \cos x$.
12. $\sin ax$.
13. $\sqrt{1+x}$.
14. $\sqrt{1-x}$.
15. $\dfrac{1}{\sqrt{1+x}}$.
16. $\dfrac{1}{1+x}$.
17. $\dfrac{1+x}{1-x}$.
18. $(1-2x)^{-\frac{1}{2}}$.
19. $\sqrt{\left(3-\frac{1}{2}x\right)^3}$.
20. $(x^2-1)^{-1}$.
21. $(2-x^3)^{\frac{4}{7}}$.
22. $(\sqrt{2}-\sqrt{x})^{-\frac{1}{3}}$.
23. $\left(\frac{1}{2}x+\frac{2}{3}\right)^{\frac{3}{2}}$.
24. $\left(\frac{1}{\sqrt{3}}-\frac{x^2}{\sqrt{2}}\right)^{\frac{7}{2}}$.
25. $(2a^{\frac{1}{4}}+3x^{\frac{2}{3}})^{-\frac{4}{3}}$.
26. $(a^{\frac{1}{5}}+3x^4)^{\frac{2}{5}}$.

By use of the binomial theorem calculate to three decimal places inclusive the values of:

27. $\sqrt{10}$.
28. $\sqrt[3]{30}$.
29. $\sqrt[4]{68}$.
30. $\sqrt[5]{1121}$.
31. $\sqrt{0.096}$.
32. $\sqrt[3]{80^2}$.
33. $\sqrt[4]{624.5}$.

Calculate to five decimal places inclusive the values of:

34. $\sin 25°$.
35. $\sin 5°$.
36. $\sin 1°$.
37. $\sin 10'$.
38. $\cos 50°$.
39. $\cos 100°$.
40. $\dfrac{1}{e}$.
41. e^{-2}.
42. $\dfrac{1}{\sqrt{e}}$.
43. $\log 1.1$.
44. $\log 1.2$.
45. $\log (.75)$.

CHAPTER XIV

Computation. Approximations. Differences and Interpolation

223. Remarks on Computation. — (1) In a series of similar computations, perform similar operations together. If the same number is to be added to each of several others write it on the edge of a slip of paper and hold it over or under each number in turn.

(2) When a result is wanted to say three decimals, computations should be carried to four places so as to avoid accumulation of errors which would vitiate the third place.

(3) As a general rule, 4-, 5-, 6-, and 7-place logarithm tables will yield respectively not more than 4, 5 6, or 7 significant figures of a number.

(4) Results should be stated with an accuracy commensurate with that of the data. Thus, if a line be measured 10 times to 0.01 ft., the mean of the 10 measures should be given to 0.001 ft. More than three places in the mean would be a useless refinement. Do not state an angle to seconds when it results from computations which render even the minute uncertain.

224. Useful Approximations. — Let the student verify that, when x, y, u, v are small decimals, we have *approximately:*

1. $(1 + x)(1 + y) = 1 + x + y.$

2. $(1 + x)(1 - y) = 1 + x - y.$

3. $(1 - x)(1 - y) = 1 - x - y.$

4. $\dfrac{1}{1 + x} = 1 - x.$

5. $\dfrac{1}{1 - x} = 1 + x.$

6. $\dfrac{1 + x}{1 + y} = 1 + x - y.$

7. $\dfrac{(1 + x)(1 + y)\cdots}{(1 + u)(1 + v)\cdots} = 1 + x + y + \cdots - u - v - \cdots.$

8. $(1 + x)^n = 1 + nx.$

As special cases of (8) we have

9. $\sqrt{1 + x} = 1 + \frac{1}{2}x.$

10. $\sqrt{1 - x} = 1 - \frac{1}{2}x.$

11. $\dfrac{1}{\sqrt{1 + x}} = 1 - \dfrac{1}{2}x.$

12. $\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x.$

13. $(1+x)^2 = 1 + 2x.$

14. $(1-x)^2 = 1 - 2x.$

15. $e^x = 1 + x.$

16. $\log_e(1+x) = x.$

17. $\log_{10}(1+x) = .43\,x.$

18. $\sin x = x$ (radians).

19. $\tan x = x.$

20. $\cos x = 1.$

More accurately:

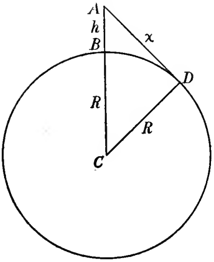
21. $\sin x = x - \frac{1}{6}x^3.$

22. $\tan x = x + \frac{1}{3}x^3.$

23. $\cos x = 1 - \frac{1}{2}x^2.$

Examples.

1. $.987 \times .993 = (1 - .013)(1 - .007) = 1 - .013 - .007 = .980.$



The error is $.013 \times .007 = .000091.$

2. $\frac{1}{.987} = \frac{1}{1-.013} = 1 + .013 = 1.013.$

3. $\sqrt{.987} = (1 - .013)^{\frac{1}{2}} = 1 - \frac{1}{2}(.013)$ $= .9935$, correct to four places.

4. Find the range of vision from a point h ft. above the surface of the earth.

Let A be the station of observation (figure),

$$AB = h \text{ ft.}, \quad BC = DC = R = 3960 \text{ miles}.$$

Then

$$R = 3960 \times 5280 \text{ ft}.$$

$$x = \sqrt{(R+h)^2 - R^2} = \sqrt{2Rh + h^2} = \sqrt{2Rh}\sqrt{1 + \frac{h}{2R}}.$$

For moderate elevations, $\frac{h}{2R}$ is small and the second radical $= 1$ approximately.

Hence $$x = \sqrt{2Rh} \text{ approximately}.$$

The error in this value of x is $\frac{h}{4R}x$ approximately.

Exercises. Calculate the approximate values of,

1. $\frac{.965}{.982}$; **2.** $\frac{.85 \times 1.12}{1.15 \times .92}$; **3.** $\sqrt{1.20}$;

4. $\frac{1}{1.125}$; **5.** $\frac{1}{\sqrt{.975}}$; **6.** $(1.15)^2$.

7. Prove the last statement of example 4.

8. How far can an observer see from a mountain one mile high?

9. What is the distance to the horizon as seen by an observer on the seashore with his eye 6 ft. above the water level? (Three-mile limit.)

10. If the range of a gun on a warship is 10 miles, how high should the lookout be stationed to detect objects coming within range?

11. What is the error in each of the approximations (1) . . . (23) when $x, y, u, v = 0.1$? When $x, y, u, v = 0.01$?

12. Calculate to four decimal places sin 130° and cos (− 100°). (Reduce to functions of angles < 45°.)

13. Calculate a 4-place table of natural sines, from 0° to 45°, at intervals of 5°.

14. As in exercise 13 for a table of natural cosines.

225. Computation of Natural Logarithms.

We have $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$

Replace x by $-x$:

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots.$$

Hence, $\log(1+x) - \log(1-x) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right],$

provided $-1 < x < 1$.

But $\log(1+x) - \log(1-x) = \log\frac{1+x}{1-x}.$

Let $\frac{1+x}{1-x} = \frac{n+1}{n}$; or, $x = \frac{1}{2n+1}.$

Then $\log(1+x) - \log(1-x) = \log(n+1) - \log n$

and

$$\log(n+1) = \log n + 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right].$$

By means of this equation $\log(n+1)$ can be calculated when $\log n$ is known. The series on the right converges rapidly and for all positive values of n. Putting successively $n = 1, 2, 3, \ldots,$ we obtain in turn log 2, log 3, log 4, ...

We will now obtain an estimate of the maximum error made in stopping at any term of the series.

Let $k = 2n + 1.$

Then the mth term of the series is

$$u_m = \frac{1}{(2m-1)k^{2m-1}},$$

and the remainder of the series will be

$$R_m = \frac{1}{(2m+1)k^{2m+1}} + \frac{1}{(2m+3)k^{2m+3}} + \frac{1}{(2m+5)k^{2m+5}} + \cdots.$$

Then R_m is certainly less, term for term, than the series

$$\frac{1}{(2m+1)k^{2m+1}}\left[1+\frac{1}{k^2}+\frac{1}{k^4}+\cdots\right]=\frac{1}{(2m+1)k^{2m+1}}\,\frac{1}{1-\frac{1}{k^2}},$$

since the series between the brackets is an infinite G. P. with ratio $\frac{1}{k^2}$. Also, since

$k = 2n + 1$ and $n \geqq 1$, $\therefore$ $k > 2$ for all values of n. Hence

$$\frac{1}{1-\frac{1}{k^2}} < 2$$

and therefore

$$R_m < \frac{2}{(2m+1)k^{2m+1}} = \frac{2}{(2m+1)(2n+1)^{2m+1}}.$$

If we now include the factor 2 which stands before the bracket in the equation giving $\log(n+1)$, the total error is less than

$$\frac{4}{(2m+1)(2n+1)^{2m+1}}$$

when $\log(n+1)$ is calculated by using only the first m terms of the series.

Thus in calculating log 5, we have $n = 5$ and the error in stopping with the mth term is less than

$$\frac{4}{(2m+1)\,11^{2m+1}}.$$

Hence when $m = 1$, the error is less than $\frac{4}{3\cdot 11^3}$; that is, if we use only the first term of the series, log 5 will come out correct to 3 decimal places inclusive. When $m = 2$, the error is less than $\frac{4}{5\cdot 11^5}$, so that the first two terms will give log 5 correct to 5 places, and so on.

Exercises.

1. What is the error in log 7 when only one term of the series is used? When two terms are used?

2. How many terms of the series are required to give log 7 correct to 10 places?

3. How many terms of the series are required to give log 17 to 20 places?

4. Calculate a four-place table of natural logarithms of the numbers from 1 to 20 inclusive.

226. Common Logarithms. — When the natural logarithm of a number is known, its common logarithm may be found by multiplying by a certain constant factor called the *modulus of the common system of logarithms*. We shall show that this modulus, or multiplier, is

$$M = \log_{10} e = 0.4342945 \ldots .$$

Let the natural logarithm of any number be x, its common logarithm y. To express y in terms of x. We have, if n be the number,

$$\log_e n = x \quad \text{and} \quad \log_{10} n = y,$$

or,

$$n = e^x \quad \text{and} \quad n = 10^y.$$

Hence

$$10^y = e^x.$$

To solve for y, take logarithms of both members to the base 10. Then

$$y = x \log_{10} e,$$

which proves our statement. To find the value of $\log_{10} e$, we need only calculate $\log_e 10$ and take the reciprocal of the result.

Exercises.

1. Calculate the modulus M to 5 places.

2. Calculate $\log_{10} 101$ to 10 places.

3. Calculate $\log_{10} 11$ to 10 places.

4. Calculate a four-place table of common logarithms of the numbers from 1 to 20 inclusive.

227. Differences. — Consider a sequence of quantities u_0, u_1, $u_2, \ldots, u_n, \ldots$, and form the differences, $\Delta u_0 = u_1 - u_0$, $\Delta u_1 = u_2 - u_1, \ldots, \Delta u_{n-1} = u_n - u_{n-1}, \ldots$, called the *first differences*. Form next the differences of these differences, called the *second differences* of the original sequence, and so on. We obtain in this way the entries in the following difference table, where the successive difference columns are denoted by Δ_1, Δ_2, Δ_3, ... and the original sequence by Δ_0.

Δ_0	Δ_1	Δ_2	Δ_3	...
u_0				
	$u_1 - u_0$			
u_1		$u_2 - 2u_1 + u_0$		
	$u_2 - u_1$		$u_3 - 3u_2 + 3u_1 - u_0$	
u_2		$u_3 - 2u_2 + u_1$		...
	$u_3 - u_2$			
u_3				
.	.			
.	.			
.	.			
u_{n-2}				
	$u_{n-1} - u_{n-2}$			
u_{n-1}		$u_n - 2u_{n-1} + u_{n-2}$		...
	$u_n - u_{n-1}$			
u_n				
.				
.				
.				

We observe that the coefficients follow the binomial law. Let the student prove by induction that this law is followed in all the successive difference columns.

228. The nth term of the sequence, in terms of its first term and the first terms of the first n difference columns.

Let the first term in the kth difference column be denoted by $\Delta_k u_0$. Then we have

$$\begin{aligned} u_0 &= u_0, \\ \Delta_1 u_0 &= u_1 - u_0, \\ \Delta_2 u_0 &= u_2 - 2\,u_1 + u_0, \\ \Delta_3 u_0 &= u_3 - 3\,u_2 + 3\,u_1 - u_0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Solving successively for $u_0, u_1, u_2, \ldots$, we have

$$\begin{aligned} u_0 &= u_0, \\ u_1 &= u_0 + \Delta_1 u_0, \\ u_2 &= u_0 + 2\,\Delta_1 u_0 + \Delta_2 u_0, \\ u_3 &= u_0 + 3\,\Delta_1 u_0 + 3\,\Delta_2 u_0 + \Delta_3 u_0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Here the coefficients again follow the binomial law, and there is suggested the formula

$$(1) \qquad u_n = u_0 + {}_nC_1\Delta_1 u_0 + {}_nC_2\Delta_2 u_0 + \cdots + \Delta_n u_0.$$

Assuming the formula true for u_n, we can show that it holds for u_{n+1}. For apply formula (1) to the nth term of the first order of differences, which is $u_{n+1} - u_n$. We obtain

$$u_{n+1} - u_n = \Delta_1 u_0 + {}_nC_1\Delta_2 u_0 + {}_nC_2\Delta_3 u_0 + \cdots + \Delta_{n+1} u_0.$$

Adding equation (1) to this we get

$$\begin{aligned} u_{n+1} = u_0 &+ ({}_nC_1 + 1)\,\Delta_1 u_0 + ({}_nC_2 + {}_nC_1)\,\Delta_2 u_0 \\ &+ ({}_nC_3 + {}_nC_2)\,\Delta_3 u_0 + \cdots + \Delta_{n+1} u_0. \end{aligned}$$

But

$${}_nC_1 + 1 = {}_{n+1}C_1, \quad {}_nC_2 + {}_nC_1 = {}_{n+1}C_2, \quad {}_nC_3 + {}_nC_2 = {}_{n+1}C_3, \ \ldots,$$

as is easily verified by substituting in the values of the binomial coefficients. Hence

$$u_{n+1} = u_0 + {}_{n+1}C_1\Delta_1 u_0 + {}_{n+1}C_2\Delta_2 u_0 + {}_{n+1}C_3\Delta_3 u_0 + \cdots + \Delta_{n+1} u_0.$$

Hence, if (1) holds for u_n, it also holds when n is replaced by $n + 1$, that is, for u_{n+1}. But we have shown that it holds for u_3; hence it holds for u_4, hence for u_5, and so on.

229. The sum of the first n terms of the sequence, in terms of its first term and the first terms of the first $n-1$ difference columns.

From the equations just preceding formula (1) we have, by addition,

$$\begin{aligned} u_0 &= u_0, \\ u_0 + u_1 &= 2\,u_0 + \Delta_1 u_0, \\ u_0 + u_1 + u_2 &= 3\,u_0 + 3\,\Delta_1 u_0 + \Delta_2 u_0, \\ u_0 + u_1 + u_2 + u_3 &= 4\,u_0 + 6\,\Delta_1 u_0 + 4\,\Delta_2 u_0 + \Delta_3 u_0. \end{aligned}$$

The coefficients on the right are respectively those of the expansions of $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, and $(1+x)^4$, the first term of the expansion being omitted in each case. Let s_n denote the sum of the first n terms of the sequence;

$$s_n = u_0 + u_1 + u_2 + \cdot\cdot\cdot + u_{n-1}.$$

Then by analogy with the preceding equations we assume that

(2) $$s_n = {}_nC_1 u_0 + {}_nC_2\Delta_1 u_0 + {}_nC_3\Delta_2 u_0 + {}_nC_4\Delta_3 u_0 + \cdots + \Delta_{n-1}u_0.$$

We show by induction that (2) holds for all values of n. Adding (1) of (**228**) to (2) and noting that $s_{n+1} = s_n + u_n$, we have

$$\begin{aligned} s_{n+1} &= ({}_nC_1+1)\,u_0 + ({}_nC_2 + {}_nC_1)\Delta_1 u_0 + ({}_nC_3 + {}_nC_2)\,\Delta_2 u_0 + \cdots + \Delta_n u_0 \\ &= {}_{n+1}C_1 u_0 + {}_{n+1}C_2\Delta_1 u_0 + {}_{n+1}C_3\Delta_2 u_0 + \cdot\cdot\cdot + \Delta_n u_0. \end{aligned}$$

Therefore (2) is true when n is replaced by $n+1$. But we verified above that (2) is true when $n = 4$. Hence it is true when $n = 5$, hence when $n = 6$, and so on.

When the *rth order of differences is zero,* all following orders of difference are also zero. Hence any term of the sequence and the sum of any number of terms can be expressed in terms of the first term of the sequence and the first terms of the first $r-1$ difference columns. For then formulas (1) and (2) both stop with the term involving $\Delta_{r-1}u_0$, and we have

(3) $$u_n = u_0 + {}_nC_1\Delta_1 u_0 + {}_nC_2\Delta_2 u_0 + \cdot\cdot\cdot + {}_nC_{r-1}\Delta_{r-1}u_0.$$

(4) $$S_n = {}_nC_1 u_0 + {}_nC_2\Delta_1 u_0 + {}_nC_3\Delta_2 u_0 + \cdot\cdot\cdot + {}_nC_r\Delta_{r-1}u_0.$$

Example. Find the sum of the squares of n consecutive integers beginning with 10.

$$s_n = 10^2 + 11^2 + 12^2 + \cdot\cdot\cdot + (10 + n - 1)^2.$$

Our difference table is as follows:

Δ_0	Δ_1	Δ_2	Δ_3
100			
	21		
121		2	
	23		0
144		2	
	25		0
169		2	
	27		
196			
⋮			

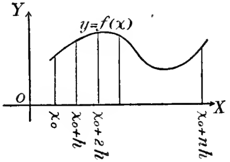
Hence $r = 3$. Then

$$S_n = {}_nC_1 u_0 + {}_nC_2 \Delta_1 u_0 + {}_nC_3 \Delta_2 u_0$$
$$= n \times 100 + \frac{n(n-1)}{1 \cdot 2} \times 21 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \times 2$$
$$= \tfrac{1}{6}(2n^3 + 57n^2 + 541n).$$

Exercises.

1. Find the sum of the squares of the integers from 1 to n inclusive.
2. Find the sum of the cubes of the integers from 1 to 20 inclusive.
3. How many balls in a square pyramid whose base has n balls on a side.
4. As in exercise 3 for a triangular pyramid.
5. Find the sum of n terms of the sequence $a, a + d, a + 2d, \ldots$.
6. Find the 10th term and the $(n + 1)$th term of the sequence 50, 72, 98, 128, 162, . . . *Ans.* 392; $2n^2 + 20n + 50$.

230. Interpolation. — Suppose the terms of the sequence $u_0, u_1, u_2, \ldots$ to be the values of a function $f(x)$ for a series of equally spaced values of x. Thus:



$$u_0 = f(x_0),$$
$$u_1 = f(x_0 + h),$$
$$u_2 = f(x_0 + 2h),$$
$$\cdots\cdots$$
$$u_n = f(x_0 + nh).$$
$$\cdots\cdots$$

These values are shown graphically in the figure, as ordinates of the curve $y = f(x)$. From the equally spaced ordinates given, we wish to calculate intermediate ones. This is called interpolation.

Replacing the u's in (1) of (**228**) by their values above, we have

$$(5) \qquad f(x_0 + nh) = f(x_0) + n\Delta_1 f(x_0) + \frac{n(n-1)}{1 \cdot 2}\Delta_2 f(x_0) + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\Delta_3 f(x_0) + \cdots.$$

This formula has been derived when n is a positive integer. It is also true for fractional values of n, provided the series on the right converges. We shall not stop for the proof, but merely give some simple applications. In practical cases the successive differences $\Delta_1 f(x_0)$, $\Delta_2 f(x_0)$, . . . become rapidly small, so that first differences are usually sufficient, second differences are occasionally needed, while third and higher differences are required only in theory or in the calculation of extensive tables.

For fractional values of n, formula (5) gives values of the function intermediate to those in the table. Thus when $n = 2\frac{1}{2}$, we get $f(x_0 + 2\frac{1}{2}h)$, which is the ordinate to the curve $y = f(x)$ falling midway between the ordinates $f(x_0 + 2h)$ and $f(x_0 + 3h)$.

Example 1. Given the values of log 100, log 101, . . . , log 109 to five decimal places, to calculate log 100.7 and log 107.35.

Here $f(x) = \log x$; $x_0 = 100$; $h = 1$. To calculate log 100.7 we put $n = .7$. Our difference table is,

$f(x)$	$\Delta_1 f(x)$	$\Delta_2 f(x)$
log 100 = 2.00000		
	+ .00432	
101 = 2.00432		− .00004
	428	
102 = 2.00860		4
	424	
103 = 2.01284		3
	421	
104 = 2.01703		5
	416	
105 = 2.02119		4
	412	
106 = 2.02531		5
	407	
107 = 2.02938		3
	404	
108 = 2.03342		3
	401	
109 = 2.03743		

Then

$$f(x_0 + nh) = \log 100.7 = \log 100 + .7 \times .00432 - \frac{.7(.7 - 1)}{1 \times 2} \times .00004 + \cdots$$

$$= 2 + .00302 + .00000 = 2.00302.$$

Here the second differences are so small that they can be neglected, and our result is that obtained by ordinary or linear interpolation. Graphically this amounts to replacing the curve $y = f(x)$ by its chords.

To calculate log 107.35, it is best to consider log 107 as the first term, or $f(x_0)$, and put $n = .35$. (We might take $f(x_0) = \log 100$ and put $n = 7.35$.) We find

$$\log 107.35 = \log 107 + .35 \times .00404 - \frac{.35\,(.35 - 1)}{1 \times 2} \times .00003 + \cdots = 2.03079.$$

Here also second differences are negligible.

All ordinary tables are constructed so that linear interpolation is sufficient.

Example 2. Given sin 10°, sin 15°, . . . , sin 45°, to calculate sin 17° 20′. The tabular numbers and their differences are given below:

$f(x)$	$\Delta_1 f(x)$	$\Delta_2 f(x)$	$\Delta_3 f(x)$
sin 10° = 0.1736			
	+ .0852		
15° = .2588		− .0020	
	832		− .0006
20° = .3420		26	
	806		6
25° = .4226		32	
	774		6
30° = .5000		38	
	736		6
35° = .5736		44	
	692		5
40° = .6428		49	
	643		
45° = .7071			

Here $x_0 = 10°$; $h = 5°$; then $17° \, 20' = x_0 + \frac{22}{15} h$ and hence $n = \frac{22}{15}$. Then

$$\sin 17° \, 20' = \sin 10° + \frac{22}{15} \times .0852 - \frac{\frac{22}{15}\left(\frac{22}{15} - 1\right)}{1 \times 2} \times .0020$$

$$- \frac{\frac{22}{15}\left(\frac{22}{15} - 1\right)\left(\frac{22}{15} - 2\right)}{1 \times 2 \times 3} \times .0006 + \cdots = .2979.$$

Here the amount contributed by the second difference is .0003, so that linear interpolation would have been inaccurate.

231. Exercises.

1. From the table of example 1 calculate log 104.6.

2. From the table of example 2 calculate sin 12° 30′, sin 27° 30′, and sin 36° 15′.

3.

n	$\frac{0.6745}{\sqrt{n(n-1)}}$
10	0.0711
15	465
20	346
25	275
30	229
35	196
40	171
45	152
50	136

Calculate the tabular number when $n = 22$; when $n = 33.6$.

4.

Altitude.	Refraction.
10°	5′ 13″.1
12°	4′ 22″.5
14°	3′ 45″.2
16°	3′ 16″.6
18°	2′ 54″.0
20°	2′ 35″.7
22°	2′ 20″.5
24°	2′ 7″.6
26°	1′ 56″.6

Calculate the refraction for altitudes 14° 40′ and 21° 25′.

5.

Greenwich mean time.	Moon's right ascension.			Moon's declination.
h	h	m	s	
0	5	14	32.14	18° 47′ 37″.7
2	5	19	49.41	18° 49′ 15″.9
4	5	25	6.62	18° 50′ 20″.6
6	5	30	23.69	18° 50′ 51″.7
8	5	35	40.59	18° 50′ 49″.4
10	5	40	57.26	18° 50′ 13″.7

Calculate the moon's right ascension and declination at $0^h\ 35^m\ 20^s$ Greenwich mean time.

6. From a four-place table take log 310, log 320, . . . , log 400. Hence calculate log 317.5.

232. Differences as a Check on Computed Values. — When a number of values of a function are calculated for equal intervals of the argument, the differences should, ordinarily, vary in a regular manner. An irregularity in one of the difference columns indicates an error in the tabular values, and often enables the computer to determine the amount of the error and so correct it.

Example.

	Δ_1	Δ_2
log 70 = 1.8451		
	+ .0300	
75 = 1.8751		− .0021
	279	
80 = 1.9030		25
	254	
85 = 1.9284		4
	258	
90 = 1.9542		23
	235	
95 = 1.9777		12
	223	
100 = 2.0000		11
	212	
105 = 2.0212		

The irregularity in Δ_2 causes us to examine Δ_1; here the differences .0254 and .0258 are probably incorrect, which throws suspicion on the tabular number standing between them, namely 1.9284. This number should evidently be larger, and by trial we find that 1.9294 is probably the correct value.

Exercises. Correct the following tables:

1.

tan 15° =	.268
16° =	.287
17° =	.306
18° =	.325
19° =	.344
20° =	.369
21° =	.384
22° =	.404
23° =	.425
24° =	.445

2.

n	$\frac{1}{n^2}$
2.0	.250
2.2	.207
2.4	.174
2.6	.158
2.8	.127
3.0	.111
3.2	.098
3.4	.087
3.6	.077

3.

Altitude.	Refraction.
10°	5′ 13″
11°	4′ 46″
12°	4′ 22″
13°	4′ 2″
14°	3′ 45″
15°	3′ 34″
16°	3′ 16″
17°	3′ 4″
18°	2′ 54″
19°	2′ 35″

CHAPTER XV

Undetermined Coefficients. Partial Fractions

233. A useful method for expanding certain expressions in series depends on the following **Theorem on Power Series.**

If the equation

$$(1) \qquad a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = 0$$

is true for all values of x from $x = 0$ to $x = x_0$ inclusive, where $x_0 \neq 0$, then all the coefficients are zero, that is,

$$a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ \ldots,\ a_n = 0,\ \ldots.$$

Proof. Since (1) is true when $x = 0$ we have, putting 0 for x, $a_0 = 0$.

Then (1) reduces to

$$a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = 0,$$

or

$$(2) \qquad x\,(a_1 + a_2x + \cdots + a^nx^{n-1} + \cdots) = 0.$$

This must be true for all values of x from 0 to x_0. Choose for x a value ε *between* 0 and x_0. Then

$$\varepsilon\,(a_1 + a_2\varepsilon + \cdots + a^n\varepsilon^{n-1} + \cdots) = 0.$$

Then, since $\varepsilon \neq 0$, we must have

$$a_1 + a_2\varepsilon + \cdots + a_n\varepsilon^{n-1} + \cdots = 0,$$

or,

$$a_1 = -\,\varepsilon\,(a_2 + a_3\varepsilon + \cdots + a_n\varepsilon^{n-2} + \cdots).$$

The series in the last parenthesis converges, and therefore has a finite sum S. For, putting $x = \varepsilon$ in (1), and omitting the first two terms, we have left the convergent series

$$a_2\varepsilon^2 + a_3\varepsilon^3 + \cdots + a_n\varepsilon^n + \cdots,$$

and this remains convergent after division by ε^2. Hence

$$a_1 = -\,\varepsilon S$$

where S depends on ε, but is finite for all values of ε between 0 and x_0. Assume now that a_1 is not equal to 0; say $a_1 = h$. We can now take ε so small that εS shall be numerically less than h; hence a_1 cannot equal h. $\therefore\ a_1 = 0$.

Then (1) reduces to

$$a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots = 0,$$

or, $$x^2\,(a_2 + a_3x + \cdots + a_nx^{n-2} + \cdots) = 0.$$

Choose for x a value ε (not necessarily the same as ε above) *between* 0 and x_0. Then

$$\varepsilon^2\,(a_2 + a_3\varepsilon + \cdots + a_n\varepsilon^{n-2} + \cdots) = 0.$$

Hence, since $\varepsilon \neq 0$, we have

$$a_2 + a_3\varepsilon + \cdots + a_n\varepsilon^{n-2} + \cdots = 0,$$

or, $$a_2 = -\,\varepsilon\,(a_3 + \cdots + a_n\varepsilon^{n-2} + \cdots) = 0.$$

Here again the series in parentheses converges and has a finite sum. Hence by taking ε sufficiently small we can show that a_2 cannot equal any number h, however small. $\therefore\ a_2 = 0$.

Similarly we show that each coefficient must be zero.

234. Theorem of Undetermined Coefficients. — If two power series in x are equal to each other for all values of x from $x = 0$ to $x = x_0$ inclusive, then the coefficients of like powers of x in the two series must be equal.

Hypothesis:

(1) $$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + \cdots \text{ when } 0 \leqq x \leqq x_0.$$

Conclusion:

$$a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2,\ \ldots,\ a_n = b_n,\ \ldots.$$

Proof. From (1), by transposition, we have

$$a_0 - b_0 + (a_1 - b_1)\,x + (a_2 - b_2)\,x^2 + \cdots + (a_n - b_n)\,x^n + \cdots = 0.$$

Hence by the preceding theorem,

$$a_0 - b_0 = 0,\ a_1 - b_1 = 0,\ a_2 - b_2 = 0,\ \ldots,\ a_n - b_n = 0.$$

Hence the conclusion stated above.

Corollary. The theorem remains true when either or both of the infinite series reduce to polynomials. We consider a polynomial of m terms as an infinite series in which all coefficients after the mth are zero.

Example 1. Develop $\frac{1-x^2}{1+x-x^2}$ into a power series.

Assume $$\frac{1-x^2}{1+x-x^2} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

Clearing, and writing the coefficients of like powers of x in vertical columns, we have

$$\begin{array}{rr|r|r|l} 1 - x^2 = a_0 + a_1 & x + a_2 & x^2 + a_3 & x^3 + \cdots \\ a_0 & + a_1 & + a_2 & & . \\ & - a_0 & - a_1 & & \end{array}$$

Equating coefficients of like powers of x, we have

$$\begin{aligned} a_0 &= 1, & \text{or,} \quad a_0 &= 1, \\ a_1 + a_0 &= 0, & a_1 &= -1, \\ a_2 + a_1 - a_0 &= -1, & a_2 &= 1, \\ a_3 + a_2 - a_1 &= 0, & a_3 &= -2. \\ \cdot \quad \cdot \quad & \cdot & \cdot \quad & \cdot \end{aligned}$$

Hence
$$\frac{1-x^2}{1+x-x^2} = 1 - x + x^2 - 2x^3 + \cdots.$$

Example 2. Develop $\frac{1+2x}{6x-5x^2+x^3}$ into a power series.

If we put
$$\frac{1+2x}{6x-5x^2+x^3} = a_0 + a_1x + a_2x^2 + \cdots,$$

clear of fractions and equate coefficients, we have to begin with $1 = 0$. This absurdity results from the fact that we have not taken a proper form for the development. By inspection we see that the quotient of $1 + 2x$ divided by $6x - 5x^2 + x^3$ should start with $\frac{1}{6x}$. To obtain the development we put

$$\frac{1+2x}{6x-5x^2+x^3} = \frac{1}{x} \cdot \frac{1+2x}{6-5x+x^2}.$$

Developing the last fraction as in example 1,

$$\frac{1+2x}{6-5x+x^2} = \frac{1}{6} + \frac{17}{36}x + \frac{79}{216}x^2 + \frac{293}{1296}x^3 + \cdots.$$

Hence
$$\frac{1+2x}{6x-5x^2+x^3} = \frac{1}{6x} + \frac{17}{36} + \frac{79}{216}x + \frac{293}{1296}x^2 + \cdots.$$

Exercises.

1. In example 1, find a_n in terms of a_{n-1} and a_{n-2}.

Find the first four terms of the expansions of:

2. $\frac{1+x}{1+x+x^2}$.

3. $\frac{1-x}{1-x-x^2}$.

4. $\frac{x}{2-x+3x^2}$.

5. $\frac{2x^2+3x}{x^2+2x+2}$.

6. $\frac{1+x^2}{1+3x+x^3}$.

7. $\frac{2-3x+x^2}{3x+4x^2-x^3}$,

235. Partial Fractions. — It is sometimes desirable to resolve a given rational fraction into a sum of simpler fractions, called *partial fractions.* This can be done when the denominator of the given fraction can be factored. Several cases arise, according to the nature of these factors.

For reasons which will presently appear, the methods to be explained apply only to fractions in which *the degree of the numerator is less than the degree of the denominator.* When this is not the case, divide numerator by denominator until a remainder of less degree than the denominator is obtained.

Case 1. The denominator can be factored into linear factors of the form $(ax + b)$, no two factors being equal.

Rule. The fraction can be resolved into a sum of simple fractions, of the form $\frac{A}{ax+b}$, equal in number to the factors of the given denominator. Here A is a constant.

Example. $$\frac{5x-1}{x^2-6x+5} = \frac{5x-1}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5}.$$

Clearing: $$5x - 1 = A(x-5) + B(x-1),$$

or, $$5x - 1 = (A+B)x - (5A+B).$$

Since the given fraction must be equal to its partial fractions for all values of x except $x = 1$ and $x = 5$, the last equation must be true for all such values of x; hence we equate coefficients of like powers of x (**233**, Corollary). We obtain

$$5 = A + B; \quad -1 = -(5A + B).$$

Hence $$A = -1; \quad B = 6.$$

$$\therefore \quad \frac{5x-1}{x^2-6x+5} = \frac{-1}{x-1} + \frac{6}{x-5}.$$

A shorter method for finding A and B is as follows: consider again the equation

$$5x - 1 = A(x-5) + B(x-1).$$

Let $x = 5$; $24 = 4B$; $B = 6$.

Let $x = 1$; $4 = -4A$; $A = -1$.

We can justify the use of the values $x = 1$ and $x = 5$, for which the given fraction and one of the partial fractions become infinite. For the equation

$$\frac{5x-1}{x^2-6x+5} = \frac{A}{x-1} + \frac{B}{x-5}$$

must hold except when $x = 1$ or $x = 5$.

Hence $$5x - 1 = A(x-5) + B(x-1)$$

is true for all values of x, except perhaps $x = 1$ and $x = 5$. It is therefore true when $x = 1 + \varepsilon$, however small ε may be; that is,

(1) $$5(1+\varepsilon) - 1 = A(1+\varepsilon-5) + B(1+\varepsilon-1).$$

Suppose our equation is not true when $x = 1$; let the two members differ by a quantity h, so that

$$5 \times 1 - 1 = A(1-5) + B(1-1) + h,$$

or,

$$4 = -4A + h.$$

From (1) we have

$$4 + \varepsilon = -4A + \varepsilon A + \varepsilon B.$$

From the last two equations, by subtraction, etc.,

$$h = \varepsilon(A + B - 1).$$

Since A and B are fixed numbers, h can be made as small as we wish by taking ε small enough. Hence h cannot equal any number except 0.

236. Case 2.—The denominator contains a linear factor repeated r times, as $(ax+b)^r$.

Rule. Corresponding to the factor $(ax+b)^r$, take a set of partial fractions of the form

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}.$$

This is the most general set of fractions having constant numerators and common denominator $(ax+b)^r$.

Example.

$$\frac{3x^2 - x + 1}{(x+2)(x-3)^3} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{D}{(x-3)^3}.$$

Clearing:

$$3x^2 - x + 1 = A(x-3)^3 + B(x+2)(x-3)^2 + C(x+2)(x-3) + D(x+2).$$

Let $x = 3$; then $25 = 5D$; $D = 5$.

Let $x = -2$; then $15 = -125A$; $A = -\frac{3}{25}$.

Since no other factors are available to furnish other values of x for substitution, we choose any convenient values, say $x = 0$ and $x = 1$.

Put $x = 0$; $1 = -27A + 18B - 6C + 2D$.

Put $x = 1$; $3 = -8A + 12B - 6C + 3D$.

Substituting the values of A and D already found, and solving for B and C, we have

$$B = \frac{3}{25};\quad C = \frac{12}{5}.$$

Hence

$$\frac{3x^2 - x + 1}{(x+2)(x-3)^3} = \frac{-3}{25(x+2)} + \frac{3}{25(x-3)} + \frac{12}{5(x-3)^2} + \frac{5}{(x-3)^3}.$$

237. Case 3. — The denominator contains a quadratic factor, $(ax^2 + bx + c)$, which cannot be resolved into real linear factors.

Rule. Corresponding to a quadratic factor $(ax^2 + bx + c)$, take a partial fraction of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

The reason for this assumption may be illustrated by a simple example.

Example. Resolve $\frac{2x - 1}{(x-1)(x^2+4)}$ into partial fractions.

If $i = \sqrt{-1}$, the factors of $x^2 + 4$ are $x + 2i$ and $x - 2i$. Suppose now we assume

$$\frac{2x - 1}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{B}{x+2i} + \frac{C}{x-2i}.$$

Combining the last two fractions into a single one, we have

$$\frac{B}{x+2i} + \frac{C}{x-2i} = \frac{(B+C)x + 2(C-B)i}{x^2+4}.$$

If now we introduce two new constants M, N in place of B, C, by the relations

$$B = M + iN; \quad C = M - iN,$$

we have

$$B + C = 2M; \quad i(C - B) = -2i^2N = 2N.$$

Hence in place of the fractions

$$\frac{B}{x+2i} + \frac{C}{x-2i},$$

where B and C involve i, we take the single fraction

$$\frac{Mx + 4N}{x^2+4},$$

where M and N are real. Then, using B in place of M and C in place of $4N$, let

$$\frac{2x - 1}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+4}.$$

Clearing: $\quad 2x - 1 = A(x^2+4) + (Bx + C)(x - 1).$

Put $\quad x = 1;$ then $\quad 1 = 5A; \quad A = \frac{1}{5}.$

Put $\quad x = 0;$ then $\quad -1 = 4A - C; \; C = \frac{9}{5}.$

Equate coefficients of x^2; then $\quad 0 = A + B; \; B = -A = -\frac{1}{5}.$

Hence

$$\frac{2x - 1}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} + \frac{-x + 9}{5(x^2+4)}.$$

238. Case 4. — The denominator contains a repeated quadratic factor, $(ax^2 + bx + c)^r$.

Rule. Corresponding to a repeated quadratic factor $(ax^2 + bx + c)^r$, take the partial fractions,

$$\frac{B_1x + C_1}{(ax^2 + bx + c)} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_rx + C_r}{(ax^2 + bx + c)^r}.$$

Example.

$$\frac{10x^3 + 7x + 4}{(x - 2)(x^2 + 3)^2} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 3} + \frac{Dx + E}{(x^2 + 3)^2}.$$

Clearing:

$$10x^3 + 7x + 4 = A(x^2 + 3)^2 + (Bx + C)(x - 2)(x^2 + 3) + (Dx + E)(x - 2).$$

Put $x = 2;\quad 98 = 49A;\quad A = 2.$

Equate coefficients of x^4, x^3, x^2, and x^0:

$$\begin{aligned} 0 &= A + B, \\ 10 &= C - 2B, \\ 0 &= 6A + 3B - 2C + D, \\ 4 &= 9A - 6C - 2E. \end{aligned}$$

Hence, $B = -2,\ C = 6,\ D = 6,\ E = -11.$

Therefore,

$$\frac{10x^3 + 7x + 4}{(x - 2)(x^2 + 3)^2} = \frac{2}{x - 2} + \frac{-2x + 6}{x^2 + 3} + \frac{6x - 11}{(x^2 + 3)^2}.$$

239. Exercises. Resolve into partial fractions:

1. $\frac{1}{3x^2 + 10x + 3}$.
2. $\frac{3x - 1}{x^2 + x - 6}$.
3. $\frac{x^2 + 6x - 8}{x^3 - 4x}$.
4. $\frac{1 + x^2}{x - x^3}$.
5. $\frac{x}{x^2 - 4x + 1}$.
6. $\frac{x^4}{x^3 + 2x^2 - x - 2}$.
7. $\frac{1}{x^4 - 1}$.
8. $\frac{x^5 + x^4 - 8}{x^3 - 4x}$.
9. $\frac{5x + 12}{x^3 + 4x}$.
10. $\frac{1}{(x^2 - 1)^2}$.
11. $\frac{x^3 - 1}{x^3 + 3x}$.
12. $\frac{x^3 + 1}{x(x - 1)^3}$.
13. $\frac{x - 8}{x^3 - 4x^2 + 4x}$.
14. $\frac{1}{x^4 + x^3 + x^2 + x}$.
15. $\frac{1}{x^3 + 1}$.
16. $\frac{x^2 - 1}{x^2 - 4}$.
17. $\frac{x^2 - 3}{x^3 - 7x + 6}$.
18. $\frac{x^5 - 2x + 1}{x^4 + 2x^3 + x^2}$.
19. $\frac{3x^2 - 2x}{x^3 - 3x^2 + 2x}$.
20. $\frac{x^2 + 3x + 4}{x^3 + 2x^2 + x}$.
21. $\frac{x^2 + 8x + 4}{x^3 + x^2 - 4x - 4}$.
22. $\frac{x^2 - 2x - 1}{x^4 + 3x^2 + 2}$.

CHAPTER XVI

DETERMINANTS

240. Determinants of the Second Order. — When two simultaneous linear equations

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

are solved for x and y, we find

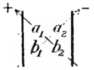
$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}; \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

To express these results it is convenient to use the notation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = (a_1b_2 - a_2b_1),$$

where the square array between vertical bars is simply another way of writing the expression forming the right member of the equation. It is called a determinant, and in particular, a determinant of the second order, because there are two rows and two columns. The quantities a_1, b_1, a_2, b_2, are called the elements of the determinant.

The value of a determinant of the second order may be obtained by forming the products of elements which constitute the diagonals of the array and giving these products the signs indicated in the scheme below:



This process is called "expanding the determinant."

The above values of x and y may now be written in the forms,

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Exercises.

1. State a rule for writing the above values of x and y.

Solve for x and y, by aid of determinants:

2. $x - y = 1,$
$2x + y = 3.$

3. $4x - 3y = 5,$
$2x + y = 1.$

4. $8x + 5y - 6 = 0,$
$4x + y + 4 = 0.$

5. $2x + y + 1 = 0,$
$6x + 3y + 2 = 0.$

6. $2x + y + 1 = 0,$
$6x + 3y + 3 = 0.$

241. Determinants of the Third Order. — We shall now define a determinant of the third order in terms of determinants of the second order by the following equation:

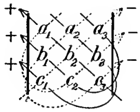
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix},$$

where the determinants on the right are to be expanded and the results multiplied by the quantity written in front of the determinants respectively.

On performing these operations and collecting terms, we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

This is the expanded form of a determinant of the third order, and may be written out by forming the products of the terms joined by arrows in the scheme below, each product to be given the sign indicated.



We may now verify by direct calculation that the values of x, y, z, obtained by solving the linear equations

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3,$$

are,

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Exercises.

1. Verify the last statement.

2. State a rule for solving three equations of the form just considered.

Solve the following systems of equations:

3. $x - y + z = 1,\quad x + 2y + 3z = 2,\quad 3x + 2y + z = 3.$

4. $2x + 2y - z = 2,\quad x + y - 2z = 1,\quad x - y + z = 4.$

5. $5x + 6y - 3z = 4,\quad 4x - 5y + 2z = 3,\quad 2x - 3y + z = 1.$

6. $3x - 6y + 9z = 2,\quad x + y + z = 1,\quad x - 2y + 3z = 2.$

7. $x - y + z = 2,\quad 2x + y + 3z = 1,\quad 2x - 2y + 2z = 4.$

8. $2x - y + 2z = 2,\quad x - 2y + 4z = 3,\quad 3x - 3y + 6z = 1.$

9. Show that a determinant of second or third order vanishes when the elements of a row or column are equal respectively to those of another row or column.

10. Show that a determinant changes sign when the signs of all the elements of any row or column are changed.

11. Show that, if the elements of any row or column be multiplied by a factor k, the determinant is multiplied by k.

242. Inconsistent or Non-independent Linear Equations. — Consider the equations

$$a_1x + b_1y = c_1 \quad \text{and} \quad ka_1x + kb_1y = c_2.$$

These are inconsistent if $c_2 \neq kc_1$; they are dependent if $c_2 = kc_1$, since in this case the second equation is k times the first.

In either case the determinant of the coefficients of x and y is 0. On solving by the determinant method, we find

$x = \infty$ and $y = \infty$, when the equations are inconsistent;

$x = \dfrac{0}{0}$ and $y = \dfrac{0}{0}$, when the equations are dependent.

That is, the inconsistent equations have no (finite) solution, while the solution is indeterminate in case of dependent equations.

Geometrically, the equations represent two straight lines which are parallel, and distinct if $c_2 \neq c_1k$; they coincide if $c_2 = c_1k$.

Hence the infinite values of x and y above are equivalent to the statement, "Parallel lines meet at infinity." In the second case, when the lines coincide, the coördinates of any point on either line satisfy both equations. Hence there are an infinite number of solutions, and hence x and y appear above as indeterminate forms. [See exercises 5 and 6 of (**240**).]

Exercises. **1.** Consider the equations

$$a_1x + b_1y + c_1z = d_1, \quad ka_1x + kb_1y + kc_1z = d_2, \quad a_3x + b_3y + c_3z = d_3.$$

The first two are inconsistent if $d_2 \neq kd_1$, and dependent when $d_1 = kd_1$. Show that in the first case the only possible solutions of the three equations are infinite, and in the second case there is an infinite number of solutions.

2. Show that the equations

$$a_1x + b_1y = 0 \quad \text{and} \quad a_2x + b_2y = 0$$

have one solution (0, 0), or an infinite number of solutions, according as the determinant of the coefficients is different from or equal to 0. Discuss also geometrically.

3. Show that the equations

$$a_1x + b_1y + c_1z = 0, \quad a_2x + b_2y + c_2z = 0, \quad a_3x + b_3y + c_3z = 0$$

have one solution (0, 0, 0), or an infinite number of solutions, according as the determinant of the coefficients is different from or equal to zero.

(*Hint.* Eliminate z so as to get two equations in x and y and discuss these as in exercise 2.)

4. Show that the equations

$$2x - 3y + 5z = 0, \quad x + y - z = 0, \quad 3x - 7y + 11z = 0$$

are not independent. What is the relation between them?

(*Hint.* To find the relation between the equations, find k_1 and k_2 such that k_1 times the first trinomial plus k_2 times the second shall equal the third.)

243. General Definition of a Determinant. — The array of n rows and n columns,

$$\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \\ c_1 & c_2 & c_3 & \cdots & c_n \\ \cdot & \cdot & \cdot & & \cdot \\ l_1 & l_2 & l_3 & \cdots & l_n \end{vmatrix}$$

is called a determinant of order n. The quantities forming the array are called the elements of the determinant.

If we form all possible products of n elements, each product to contain one and only one element from each row and column, and if these products are given proper signs, as will presently be indicated, and added algebraically, the sum so obtained is defined to be the value of the determinant.

Each product of n elements so obtained is called a term of the expanded form of the determinant.

The elements $a_1, b_2, c_3, \ldots, l_n$ form the principal diagonal.

The term $a_1b_2c_3 \ldots l_n$ is called the principal term of the expansion.

244. *Every term of the expansion of the determinant can be formed from the principal term by rearranging the subscripts, leaving the letters in their natural order.*

For every term contains all the letters and all the subscripts, and each only once, since it is a product containing one and only one element from each row and each column. Hence if the letters in any term be arranged in their natural order, the subscripts will form some arrangement of the numbers $1, 2, 3, \ldots, n$.

Conversely, every rearrangement of subscripts in the principal term, the letters being left in their natural order, yields a term of the expansion, since it contains one element and only one from each row and each column.

Therefore all the terms of the expansion can be obtained by forming all possible arrangements of subscripts in the principal term.

We shall use the symbol Δ_n to indicate our determinant of order n. Then we can write the equation

$$\Delta_n = \Sigma \pm a_1b_2c_3 \ldots l_n, \quad (\Sigma = \text{sigma})$$

where the symbol Σ (sign for a sum) means that we are to form the algebraic sum of all terms which may be formed from the term written by forming all possible arrangements of the subscripts; the signs of the terms so formed remain to be determined.

245. Number of Terms in the Expansion of Δ_n. — The number of terms in the expansion of a determinant of order n is $1 \times 2 \times 3 \times \cdots \times n$, or $\underline{|n}$.

Proof. We need only show that the number of possible arrangements of the subscripts $1, 2, 3, \ldots n$, is $\underline{|n}$.

Starting with the natural order, and interchanging 1 in turn with 2, 3, . . . , n, we form the n arrangements

$$\begin{array}{l} 1\ 2\ 3\ \ldots\ n, \\ 2\ 1\ 3\ \ldots\ n, \\ 2\ 3\ 1\ \ldots\ n, \\ \ldots\ldots \\ 2\ 3\ 4\ \ldots\ 1. \end{array}$$

In any one of these, keep 1 fixed in its position, and interchange 2 with 3, 4, . . . , n. In this way we form $n - 1$ arrangements in which 1 occupies a given place. Treating each of the n arrangements written above similarly, we obtain altogether $n(n - 1)$ arrangements. Each of these gives rise to a group of $n - 2$ arrangements, including itself, by interchanging 3 with 4, 5, . . . , n. Hence we obtain $n(n-1)(n - 2)$ arrangements. Proceeding similarly we find the total number of arrangements to be $\underline{|n}$.

246. Signs of the Terms in the Expansion of Δ_n.

Inversion. An arrangement of the numbers 1, 2, 3, . . . , n is called an inversion. An inversion is even or odd according as the number of times a greater number precedes a lesser number is even or odd.

Thus, the possible inversions of 3 numbers are

$$123,\ 213,\ 231,\ 321,\ 312,\ 132;$$

of these the first, third, and fifth are even, the others odd.

Further, the inversion of the subscripts in the term $a_4b_2c_3d_1$ is even. For 4 precedes 2, 3, and 1, and 3 precedes 1, making a greater subscript precede a lesser one 4 times.

We now define the sign of each term of the expansion of Δ_n by the rule that the sign shall be plus when the inversion of the subscripts is even, minus when the inversion is odd.

Our determinant is now completely defined.

Exercise. Write out the expansion of

$$\Delta_4 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

247. Properties of Determinants.

1. *A determinant is unchanged in value when its rows and columns are interchanged.*

For the expansion remains unaltered.

2. *Interchanging two adjacent rows or columns changes the sign of the determinant.*

For each term of the expansion will change sign, since two adjacent subscripts will be interchanged; hence even inversions change to odd, and vice versa.

By repeated application of this rule it follows that *if any two rows or any two columns be interchanged, the sign of the determinant changes.*

3. *If all the elements of a row or column are* 0, *the determinant* $= 0$.

For each term of the expansion contains a zero factor.

4. *When all the elements of a row, or column, contain a common factor, this may be taken out and written as a factor of the whole determinant.*

For each term of the expansion will contain this factor.

It follows that, *to multiply a determinant by any factor, we need only multiply the elements of any row or column by this factor.*

5. *If two rows or columns are alike, the determinant* $= 0$.

For by interchanging them we would have $\Delta_n = -\Delta_n$; $\therefore \Delta_n = 0$.

6. *If the elements of two rows or columns differ only by a common factor, the determinant* $= 0$.

For by taking out the common factor the two rows or columns become equal.

7. *If in the expansion of Δ_n we collect the terms which contain the several elements of any row or column, say the jth row, we have*

$$\Delta_n = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ j_1 & j_2 & j_3 & \cdots & j_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & l_3 & \cdots & l_n \end{vmatrix} = j_1J_1 + j_2J_2 + \cdots j_nJ_n.$$

Here J_1 is called the cofactor of the element j_1, and similarly for $J_2, \ldots, J_n$.

8. *A determinant is unaltered in value when the elements of any row are increased by a constant multiple of the corresponding elements of another row. Similarly for columns.*

For suppose that we add to the elements of the first row k times the elements of the second. We obtain the determinant

$$\Delta_n' = \begin{vmatrix} a_1 + kb_1, & a_2 + kb_2, & \ldots, & a_n + kb_n \\ b_1 & b_2 & \ldots & b_n \\ c_1 & c_2 & \ldots & c_n \\ \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & \ldots & l_n \end{vmatrix}$$

Let $A_1, A_2, \ldots, A_n$ be the cofactors of the elements of the first row, so that

$$\begin{aligned}\Delta_n' &= (a_1 + kb_1)A_1 + (a_2 + kb_2)A_2 + \cdots + (a_n + kb_n)A_n \\ &= (a_1A_1 + a_2A_2 + \cdots + a_nA_n) + \\ &\qquad k(b_1A_1 + b_2A_2 + \cdots + b_nA_n). \\ &= \begin{vmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & \ldots & l_n \end{vmatrix} + k \begin{vmatrix} b_1 & b_2 & \ldots & b_n \\ b_1 & b_2 & \ldots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & \ldots & l_n \end{vmatrix}.\end{aligned}$$

The first of these determinants is Δ_n, the second equals 0.

$$\therefore \qquad \Delta_n' = \Delta_n.$$

It follows that we can add to the elements of any row any linear combination of corresponding elements of other rows.

Example. Without expanding, show that

$$\begin{vmatrix} 102 & 104 & 106 \\ 99 & 98 & 97 \\ 1 & 2 & 3 \end{vmatrix} = 0.$$

Subtract the second row from the first. The new form is

$$\begin{vmatrix} 3 & 6 & 9 \\ 99 & 98 & 97 \\ 1 & 2 & 3 \end{vmatrix}.$$

This is zero, by **6.**

9. *If the cofactors of any row or column be multiplied by the elements of any other row or column, the sum of the products is zero.*

For we have

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & \cdots & l_n \end{vmatrix} = a_1A_1 + a_2A_2 + \cdots + a_nA_n.$$

Replace the a's by the elements of any other row, as the second. The result is

$$\begin{vmatrix} b_1 & b_2 & \cdots & b_n \\ b_1 & b_2 & \cdots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & \cdots & l_n \end{vmatrix} = b_1A_1 + b_2A_2 + \cdots + b_nA_n = 0.$$

10. *If we strike out from Δ_n the jth row and kth column, the remaining determinant, of order $n-1$, is designated by $\Delta_{j,k}$, and is called the minor of the element standing at the intersection of the row and the column struck out.*

Thus the minors of a_1, a_2, and a_3 in the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

are, respectively,

$$\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, \quad \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

We shall now show that, *except as to sign, the minor of any element equals the cofactor of that element.* We shall consider a determinant of third order, although the argument will apply to determinants of any order. We have

$$\Delta_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1A_1 + a_2A_2 + a_3A_3,$$

where A_1, A_2, A_3 are the cofactors of a_1, a_2, a_3, respectively.

Then $$A_1 = \Delta_{1,1}.$$

For, since a_1A_1 contains all the terms of Δ_3 which involve a_1, and since the expansion of $\Delta_{1,1}$ contains all possible products of elements, one from each column and each row except the first, therefore A_1 and $\Delta_{1,1}$ must be identical. Now interchange the first two columns, so that Δ_3 becomes $-\Delta_3$. Then

$$-\Delta_3 = \begin{vmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{vmatrix} = -a_2A_2 - a_1A_1 - a_3A_3.$$

The minor of a_2 is unchanged, namely $\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$. The expansion of this multiplied by a_2 gives all the terms of the expansion of $-\Delta_3$ containing a_2. But these are also contained in $-a_2A_2$. Hence $\Delta_{1,2} = -A_2$, or $A_2 = -\Delta_{1,2}$.

In the same way, by moving the third column into first place by two successive interchanges, which does not alter the sign of the determinant, we find $\Delta_{1,3} = A_3$.

Let $A_{j,k}$ denote the cofactor of the element standing at the intersection of the jth row and kth column of Δ_n; we can bring this element to the intersection of the first row and column by $j - 1 + k - 1$ successive interchanges of rows and columns. Hence Δ_n will become $(-1)^{j+k-2} \cdot \Delta_n$ or $(-1)^{j+k}\Delta_n$, since $(-1)^{-2} = 1$; hence by reasoning as above we find

$$A_{j,k} = (-1)^{j+k}\Delta_{j,k}.$$

11. *We can now expand Δ_n according to the elements of its first row in the form*

$$\Delta_n = a_1\Delta_{1,1} - a_2\Delta_{1,2} + a_3\Delta_{1,3} - a_4\Delta_{1,4} + \dots + (-1)^{n-1}\Delta_{1,n}.$$

To expand Δ_n according to the elements of any other row, we can move this row into first place and then apply the last formula.

By this rule we can express a determinant of order n in terms of determinants of order $n - 1$. Hence by repeated application of the rule we can write out the complete expansion.

By a similar process the determinant can be expanded according to the elements of any column.

248. Solution of Systems of Linear Equations. — We shall illustrate the method of solving a system of n linear equations involving n unknowns by considering three such equations with three unknowns.

Solve for x, y, and z the system of equations

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3.$$

Let the determinant of the coefficients be denoted by Δ, so that

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Let the cofactors of a_1, a_2, a_3 be A_1, A_2, A_3 respectively.

Multiply the given equations in order by A_1, A_2, A_3, and add the results. We obtain

$$(a_1A_1 + a_2A_2 + a_3A_3)\,x + (b_1A_1 + b_2A_2 + b_3A_3)\,y + (c_1A_1 + c_2A_2 + c_3A_3)\,z = d_1A_1 + d_2A_2 + d_3A_3.$$

From (7) and (9) of (**247**) we see that the coefficient of x is Δ, and of y and z zero. Hence we get

$$x = \frac{d_1A_1 + d_2A_2 + d_3A_3}{\Delta} = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Similarly by multiplying by the cofactors of b_1, b_2, b_3 and adding we get y, and by multiplying by the cofactors of c_1, c_2, c_3 and adding we get z. The results are as given in (**241**).

In precisely the same way we can solve n linear equations in n unknowns.

The exceptional cases which arise when Δ, the determinant of the coefficients, is zero, have been considered in (**242**) for the case of two and three equations. A similar discussion applies to the case of n equations.

When the equations are homogeneous (i.e., $d_1 = 0$, $d_2 = 0$, $d_3 = 0 \ldots$), and $\Delta \neq 0$, the only solution is $x = 0$, $y = 0$, $z = 0$, $\ldots$; when $\Delta = 0$, there exists an infinite number of solutions.

249. Exercises. Evaluate the following determinants:

1. $\begin{vmatrix} a & 1 & 3 \\ a+1 & 2 & 2 \\ a+2 & 3 & 1 \end{vmatrix}$ **2.** $\begin{vmatrix} a & h & g \\ h & g & f \\ g & f & c \end{vmatrix}$ **3.** $\begin{vmatrix} o & a & b \\ -a & o & c \\ -b & -c & o \end{vmatrix}$

4. $\begin{vmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 10 \\ 0 & 3 & 2 & 4 \\ 6 & 2 & 0 & 3 \end{vmatrix}$ **5.** $\begin{vmatrix} 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 \\ 3 & 4 & -3 & 2 \\ 1 & -1 & 4 & 5 \end{vmatrix}$ **6.** $\begin{vmatrix} 6 & 2 & 8 & 2 \\ 2 & 2 & 8 & 5 \\ 1 & 6 & 4 & 2 \\ 3 & 2 & 5 & 3 \end{vmatrix}$

7. $\begin{vmatrix} 3 & 1 & 1 & 2 \\ 1 & 5 & 0 & 3 \\ 2 & -2 & 1 & 6 \\ 0 & 4 & -5 & 3 \end{vmatrix}$ **8.** $\begin{vmatrix} 2 & 5 & 4 \\ 4 & 1 & 6 \\ 1 & -3 & 7 \end{vmatrix}$ **9.** $\begin{vmatrix} 0 & 4 & 4 & 4 \\ -1 & -9 & -1 & 9 \\ -1 & 7 & 1 & -1 \\ 9 & 16 & 27 & 23 \end{vmatrix}$

10. $\begin{vmatrix} o & a & b & c \\ -a & o & d & e \\ -b & -d & o & f \\ -c & -e & -f & o \end{vmatrix}$ **11.** $\begin{vmatrix} a_1 & o & o & o \\ a_2 & b_2 & o & o \\ a_3 & b_3 & c_3 & o \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

12. Show, without expanding, that

$$\begin{vmatrix} 6 & 1 & -7 \\ 5 & -10 & 5 \\ 4 & 3 & -7 \end{vmatrix} = 0.$$

13. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (y-x)(z-x)(z-y).$$

14. Show that

$$\begin{vmatrix} 18 & 36 & 58 & 50 \\ 26 & 39 & 80 & 78 \\ 17 & 39 & 55 & 45 \\ 9 & 16 & 27 & 23 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 4 & 4 \\ -1 & -9 & -1 & 9 \\ -1 & 7 & 1 & -1 \\ 9 & 16 & 27 & 33 \end{vmatrix}$$

15. Give two pairs of values of x and y which satisfy the equation

$$\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 0.$$

16. Give the coördinates of two points on the line

$$\begin{vmatrix} x & y & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 0.$$

17. Trace the graph of

$$\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 0.$$

18. Give the coördinates of two points on the line

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0.$$

19. Give three sets of values of x, y, z which satisfy the equation

$$\begin{vmatrix} x & y & z & 1 \\ 3 & 1 & -2 & 1 \\ 1 & -2 & 2 & 1 \\ -1 & 4 & 1 & 1 \end{vmatrix} = 0.$$

20. As in 19, for the equation

$$\begin{vmatrix} x & y & z & 1 \\ a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \end{vmatrix} = 0.$$

Prove the following identities:

21. $\cos(x+y) = \begin{vmatrix} \cos x & \sin x \\ \sin y & \cos y \end{vmatrix}.$

22. $\sin(x-y) = \begin{vmatrix} \sin x & \cos x \\ \sin y & \cos y \end{vmatrix}.$

23. $\cos 2x = \begin{vmatrix} \cos x & \sin x \\ \sin x & \cos x \end{vmatrix}.$

24. $\begin{vmatrix} a & b & c \\ \sin x & \sin y & \sin z \\ \cos x & \cos y & \cos z \end{vmatrix} = a\sin(y-z) + b\sin(z-x) + c\sin(x-y).$

25. $\begin{vmatrix} \sin^2 x & \cos^2 x & 1 \\ \sin^2 y & \cos^2 y & 1 \\ \sin^2 z & \cos^2 z & 1 \end{vmatrix} = 0.$

26. $\begin{vmatrix} \cos x & \sin x\cos x & \cos x(\sin y + \sin z) \\ \cos y & \sin y\cos y & \cos y(\sin x + \sin z) \\ \cos z & \sin z\cos z & \cos z(\sin x + \sin y) \end{vmatrix} = 0.$

27. $\begin{vmatrix} \sin x & \sin 2x & \sin 3x \\ \sin^2 x & \sin^2 2x & \sin^2 3x \\ \sin 2x & \sin 4x & \sin 6x \end{vmatrix} = 2\sin x \sin 2x \sin 3x(\sin 2x - 2\sin x).$

28. Show that

$$\begin{vmatrix} a+a' & b+b' & c+c' \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} + \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & k \end{vmatrix}$$

29. Show that the equations

$$-4x + y + z = 0 \quad \text{and} \quad x - 2y + z = 0$$

are satisfied by

$$x : y : z = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} : - \begin{vmatrix} -4 & 1 \\ 1 & 1 \end{vmatrix} : \begin{vmatrix} -4 & 1 \\ 1 & -2 \end{vmatrix}.$$

30. Show that the equations $\begin{cases} lx + my + nz = 0, \\ l'x + m'y + n'z = 0, \end{cases}$

are satisfied by

$$x : y : z = \begin{vmatrix} m & n \\ m' & n' \end{vmatrix} : - \begin{vmatrix} l & n \\ l' & n' \end{vmatrix} : \begin{vmatrix} l & m \\ l' & m' \end{vmatrix}.$$

31. Show that the equations $\begin{cases} 2x + 4y + 5z = 0, \\ 3x + 5y + 6z = 0, \\ 4x + 6y + 7z = 0, \end{cases}$

are satisfied by $x : y : z = 1 : -3 : 2$.

Solve the following systems of equations:

32. $2x + 3y - 4z + 7 = 0,$
$7x - 4y - 1 = 0,$
$9x - 4z + 1 = 0.$

33. $20u + 2v - 7 = 0,$
$4v + 5w - 1 = 0,$
$4u - 3w + 2 = 0.$

34. $-r + s + t + u = 4,$
$r - s + t + u = 3,$
$r + s - t + u = 2,$
$r + s + t - u = 1.$

35. $2x - y - 3z + w = 1,$
$x + 2y + z - w = 2,$
$3x - 3y - z + 2w = -1,$
$-x - y + 2z - 3w = 0.$

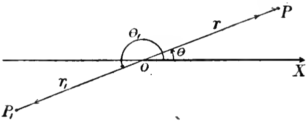
36. $k + l + m - 2n = 1,$
$2k - l + 2m - 4n = 2,$
$-k + 2l + 3m - 6n = -2,$
$k - l + 4m - 8n = -1.$

CHAPTER XVII

Polar Coördinates. Complex Numbers. De Moivre's Theorem. Exponential Values of $\sin x$ and $\cos x$. Hyperbolic Functions

250. Polar Coördinates. — We have made repeated use of the system of rectangular coördinates, in which the position of any point in the plane is defined by its abscissa and ordinate. A second system of coördinates defines the position of a point with reference to a single fixed line, called the **initial line,** and a fixed point on this line, called the **origin** or **pole.**

In the figure, let OX be the initial line and O the pole. We shall consider OX as the positive direction of the initial line. Let P be a point in the plane to be considered. The position of P is then fixed by its distance $OP = r$ from O, called the **radius vector,** and by the angle $XOP = \theta$, called the **vectorial angle**. Then r, θ are called the **polar coördinates** of P, and the point is indicated by (r, θ). Similarly P_1 is the point (r_1, θ_1). The coördinate θ is positive when measured counter-clockwise from OX; r is positive when measured from O along the terminal side of θ; it is negative when measured from O along the terminal side of θ produced back through O. Thus the points (5, 30°) and (−5, 210°) coincide. Similarly with (135°, −3) and (−45°, 3).



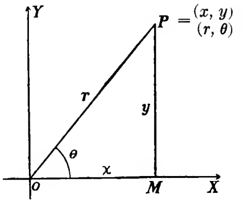
Exercise. Plot the following points:

$$(45°, 1);\ (45°, -1);\ (60°, 3);\ (-60°, 3);\ \left(\frac{\pi}{8}, 4\right);\ \left(-\frac{2\pi}{5}, 2\right);\ \left(\frac{5\pi}{6}, \frac{2}{3}\right);$$

$$\left(-\frac{5\pi}{6}, -\frac{2}{3}\right);\ \left(\frac{3\pi}{2}, 1\right);\ \left(-\frac{3\pi}{2}, -1\right);\ (800°;\ \pi).$$

Calculate the rectangular coördinates of each of these points, taking O as origin and OX as the x-axis.

251. Relation between Polar and Rectangular Coördinates. — Let O be the origin and OX the initial line of a system of polar coördinates (figure). Let OX and OY be the axes of a rectangular system of coördinates. Then



$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta; \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \tan^{-1}\dfrac{y}{x}. \end{cases}$$

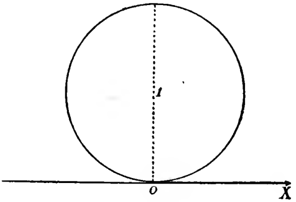
252. Curves in Polar Coördinates. — When r and θ are unrestricted, the point (r, θ) may take any position in the plane. When r and θ are connected by an equation, the point (r, θ) is in general restricted to a curve, the equation between r and θ being called the polar equation of the curve.

Example 1. Trace the curve whose polar equation is $r = \sin\theta$.

Assume a series of values for θ, calculate the corresponding values of r and plot the points whose coördinates are corresponding values of r and θ.

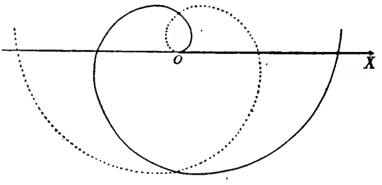
$\theta =$	0°,	30°,	60°,	90°,	120°,	150°,	180°,	210°,	240°,	270°,	300°,	330°,	360°.
$r =$	0,	.5,	.87,	1.0,	.87,	.5,	0,	−.5,	−.87,	−1.0,	−.87,	−.5,	0.

The graph is shown in the figure. For values of $\theta > 360°$, and for negative angles, no new points are obtained. The curve is a circle, with radius $= \frac{1}{2}$.



Example 2. Trace the curve $r = 2\theta$.

Here θ is understood to be in radians.



$$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \ldots \infty.$$

$$r = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \ldots \infty.$$

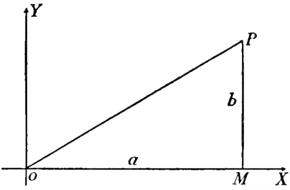
For negative values of θ we get corresponding negative values of r. The curve is the double spiral in the figure, the branches shown by the full line and the dotted line being obtained from the positive and the negative values of θ respectively.

Exercises. Trace the following curves:

1. $r = 2\sin\theta$.	**5.** $r = \sin^{-1}\theta$.	**9.** $r = e^{\theta}$.
2. $r = \cos\theta$.	**6.** $r = \tan^{-1}\theta$.	**10.** $r = \log_{10}\theta$.
3. $r = \tan\theta$.	**7.** $r\theta = 1$.	**11.** $r = 4$.
4. $r = \sec\theta$.	**8.** $r = 2^{\theta}$.	**12.** $\theta = \frac{\pi}{4}$.

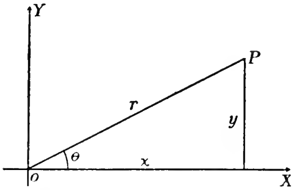
253. Complex Numbers. — Let a and b denote any two *real* numbers and $i = \sqrt{-1}$. Then the quantity $a + ib$ is called a *complex number*. It may be considered as made up of a real units and b imaginary units, $a \times 1 + b \times i$.

Real numbers can be represented by points on a straight line. To represent complex numbers geometrically, we require a plane.



Let OX and OY be a system of rectangular axes, and P a point in their plane having coördinates (a, b) (figure). Then P is called the representative point of the complex number $a + ib$.

When $b = 0$, P lies on the x-axis, and the complex number reduces to a real number. Thus all points on the x-axis correspond to real numbers, and this line is called the axis of real numbers.



Let P (figure) be a point (x, y) in the plane, and let z be the complex number represented by P. Then

$$z = x + iy.$$

Now take OX as the initial line and O as the pole of a system of polar coördinates. Let the polar coördinates of P be (r, θ). Then

$$x = r\cos\theta;\ y = r\sin\theta.$$

Hence

$$z = x + iy = r(\cos\theta + i\sin\theta).$$

Here r is called the **modulus** and θ the **angle** of the complex number z.

When r is fixed, and θ is changed by integral multiples of 2π, we obtain a set of complex numbers of the form,

$$z = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)];$$

$$n = 0, \pm 1, \pm 2, \ldots .$$

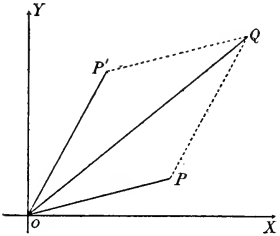
All these numbers have the same representative point.

254. Addition of Complex Numbers. — The sum of two complex numbers,

$$z = x + iy \quad \text{and} \quad z' = x' + iy',$$

we *define* by the equation

$$z + z' = (x + x') + i(y + y').$$



We proceed to consider this sum geometrically. Let P, P' (figure) be the representative points of z, z' respectively. On OP and OP' as adjacent sides construct the parallelogram $OPQP'$. Then Q is the representative point of $z + z'$. For the coördinates of Q are $(x + x', y + y')$.

The difference of the two complex numbers z and z' we may define by the equation

$$z - z' = (x - x') + i(y - y').$$

Exercise. Give a geometric construction for the representative point of $z - z'$.

255. Multiplication of Complex Numbers. — The product of the two complex numbers,

$$z = r(\cos\theta + i\sin\theta) \quad \text{and} \quad z' = r'(\cos\theta' + i\sin\theta'),$$

we *define* by the equation

$$zz' = rr'(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta').$$

Multiplying out the product of the two binomials we find

$$zz' = rr'[\cos\theta\cos\theta' - \sin\theta\sin\theta' + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')]$$
$$= rr'[\cos(\theta+\theta') + i\sin(\theta+\theta')].$$

Therefore the modulus of the product zz' equals the product of the moduli of z and z', and the angle of zz' equals the sum of the angles of z and z'.

By repeating this process we find

$$zz'z'' \cdots = rr'r'' \cdots [\cos(\theta+\theta'+\theta''+\cdots) + i\sin(\theta+\theta'+\theta''+\cdots)]$$

for any finite number of factors $z, z', z'', \ldots$.

When the factors are all equal this reduces to

$$z^n = r^n(\cos n\theta + i\sin n\theta),$$

n being a positive integer.

Exercise. Show that the above definition of the product zz' is the same as

$$zz' = xx' - yy' + i(xy' + x'y),$$

where $$z = x + iy \quad\text{and}\quad z' = x' + iy'.$$

256. De Moivre's Theorem. — When $r = 1$, then $z = \cos\theta + i\sin\theta$. Hence by the above result we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

This equation contains what is known as *De Moivre's Theorem.*

257. Definition of z^p. — Let p be any *real* number, positive or negative, rational or irrational. Then by analogy with the result for z^n when n is a positive integer, we *define* z^p by the equation

$$z^p = r^p(\cos p\theta + i\sin p\theta),$$

where $$z = r(\cos\theta + i\sin\theta).$$

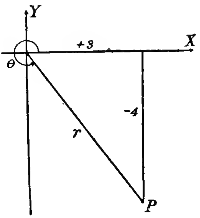
Then, if q also be real, we have

$$z^q = r^q(\cos q\theta + i\sin q\theta),$$

and

$$z^pz^q = r^{p+q}[\cos(p+q)\theta + i\sin(p+q)\theta] = z^{p+q}.$$

Hence the rules for exponents will be the same when the base is a complex number as when the base is real.



Examples.

1. Find the modulus and angle of $z = 3 - 4i$.

Here $\qquad 3 = r\cos\theta; \quad -4 = r\sin\theta.$

$$\therefore \quad r = \sqrt{3^2 + 4^2} = 5; \ \tan\theta = \frac{-4}{3},$$

or, $$\theta = \tan^{-1}\left(-\frac{4}{3}\right).$$

The angle lies in the fourth quadrant.

2. Express $2(\cos 150° - i\sin 150°)$ in the form $x + iy$.

$$2(\cos 150° - i\sin 150°) = 2\left(-\frac{1}{2}\sqrt{3} - \frac{i}{2}\right) = -\sqrt{3} - i.$$

3. Find the value of $(1 + i)^2(2 - 3i)$.

$$(1 + i)^2 = 1 + 2i + i^2 = 2i.$$

$$(1 + i)^2(2 - 3i) = 2i(2 - 3i) = 4i - 6i^2 = 6 + 4i.$$

Exercises.

1. Find the modulus and angle of

$$1 - i; \quad 4 + 3i; \quad -5 + 11i; \quad 2i; \quad 2; \quad (1+i)(1-i);$$
$$3\sqrt{3} + 3i; \quad (3\sqrt{3} - 3i)^2; \quad (1 + i\sqrt{3})(\sqrt{3} + i).$$

Give figure for each case.

2. Find the value of:

$(1+i)^3$; $(1-i)^4$; $(1+i)^2(1+2i)^2$; $(3-3i)^2(\sqrt{3}+i)^3$; $(1-i\sqrt{3})^6$.

258. Theorem. *If P and Q are any real quantities and if $P + iQ = 0$, then $P = 0$ and $Q = 0$.*

Proof. By hypothesis, $P + iQ = 0$ or $P = -iQ$.

Squaring, $$P^2 = -Q^2.$$

Now P^2 and Q^2 must be positive, hence the last equation states that a positive quantity equals a negative quantity. This is impossible unless both quantities are zero.

$$\therefore \quad P = 0 \quad \text{and} \quad Q = 0.$$

This theorem is used to replace a given equation of the form

$$P + iQ = 0$$

by the equivalent equations

$$P = 0; \quad Q = 0.$$

As a *corollary* we have, if

$$P + iQ = P' + iQ',$$

then

$$P = P' \quad \text{and} \quad Q = Q'.$$

For the given equation is equivalent to $(P - P') + i(Q - Q') = 0$.

259. The nth Roots of Unity. — To solve the equation

$$x^n - 1 = 0, \quad \text{or} \quad x^n = 1,$$

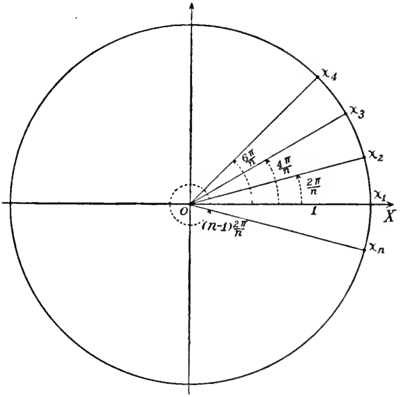
replace 1 by its value $\cos 2k\pi + i \sin 2k\pi$, k being an integer. We obtain

$$x^n = \cos 2k\pi + i \sin 2k\pi.$$

Taking the nth roots of both members we have, by putting $p = \frac{1}{n}$ in (257),

$$x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}.$$

Here k may be any integer; letting $k = 0, 1, 2, \ldots n - 1$, we obtain n distinct values of x, that is, n distinct nth roots of 1. For other values of k we obtain the same roots over again.



Geometric Representation of the nth Roots of Unity. — The nth roots of 1 are,

$$k = 0; \quad x_1 = \cos 0 + i \sin 0 = 1,$$

$$k = 1; \quad x_2 = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n},$$

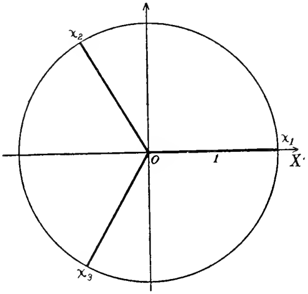
$$k = 2; \quad x_3 = \cos \frac{4\pi}{n} + i \sin \frac{4\pi}{n},$$

$$\cdots \quad \cdots \quad \cdots$$

$$k = n - 1; \quad x_n = \cos \frac{2(n-1)\pi}{n} + i \sin \frac{2(n-1)\pi}{n}.$$

The representative points of x_1, x_2, x_3, . . . x_n are obtained as n equally spaced points on a circle of radius 1, the coördinates of the first point being (1, 0) (figure).

To obtain the nth roots of any number a, we need only multiply one of its arithmetic nth root by the nth roots of unity.



Example. Find the cube roots of unity.

These are given by

$$x = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3};\quad k = 0, 1, 2.$$

$$k = 0;\quad x_1 = \cos 0° + i\sin 0° = 1.$$

$$k = 1;\quad x_2 = \cos 120° + i\sin 120° = -\frac{1}{2} + \frac{i}{2}\sqrt{3}.$$

$$k = 2;\quad x_3 = \cos 240° + i\sin 240° = -\frac{1}{2} - \frac{i}{2}\sqrt{3}.$$

To find the cube roots of 8, we have $\sqrt[3]{8} = 2\sqrt[3]{1} = 2;\ -1 + i\sqrt{3};\ -1 - i\sqrt{3}$. (We here use $\sqrt[3]{8}$ to denote *any* cube root of 8, not merely the principal root.)

Exercises.

1. Solve the equations $x^3 - 1 = 0$ and $x^3 - 8 = 0$ algebraically and compare with above results.

Solve the following equations by the trigonometric method and give a figure for each case:

2. $x^4 = 1$; **4.** $x^5 = 1$; **6.** $x^6 = 1$;
3. $x^4 = 81$; **5.** $x^5 = 32$; **7.** $x^6 = 27$.

260. To express sin $n\theta$ and cos $n\theta$ in terms of powers of sin θ and cos θ, n being a positive integer.

We have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Expand the left member by the binomial theorem, reduce all powers of i to ± 1 or $\pm i$, and group the real terms and those involving i. The above equation then becomes

$$\cos n\theta + i\sin n\theta = \left(\cos^n\theta - \frac{n(n-1)}{\lfloor 2}\cos^{n-2}\theta\sin^2\theta + \cdots\right)$$

$$+ i\left(n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{\lfloor 3}\cos^{n-3}\theta\sin^3\theta + \cdots\right).$$

This equation has the form

$$P + iQ = P' + iQ'.$$

Hence by the corollary in (258) we have

$$\cos n\theta = \cos^n \theta - \frac{n(n-1)}{\lfloor 2} \cos^{n-2}\theta \sin^2\theta + \cdots.$$

$$\sin n\theta = n \cos^{n-1}\theta \sin\theta - \frac{n(n-1)(n-2)}{\lfloor 3} \cos^{n-3}\theta \sin^3\theta + \cdots.$$

Examples.

$$\sin 4\theta = 4\cos^3\theta \sin\theta - 4\cos\theta \sin^3\theta.$$
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

Exercises. Expand in powers of $\sin\theta$ and $\cos\theta$:

1. $\sin 3\theta$; **3.** $\cos 4\theta$; **5.** $\sin 6\theta$;

2. $\cos 3\theta$; **4.** $\sin 5\theta$; **6.** $\cos 7\theta$.

261. Exponential Values of sin x and cos x.—We have the expansions, (**219**),

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots;$$

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots,$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots.$$

In the first series replace x by ix and *define* the result to be e^{ix}; noting that

$$i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ \cdots,$$

we obtain

$$e^{ix} = 1 + ix - \frac{x^2}{\lfloor 2} - i\frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + i\frac{x^5}{\lfloor 5} - \cdots,$$

$$= \left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots\right) + i\left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots\right).$$

Hence

$$e^{ix} = \cos x + i \sin x.$$

Replacing x by $-x$;

$$e^{-ix} = \cos x - i \sin x.$$

From these equations we find

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}; \quad \sin x = \frac{e^{ix} - e^{-ix}}{2\,i}.$$

These formulas are useful in many applications of the trigonometric functions.

Exercises. Using the exponential values of $\sin x$ and $\cos x$, show that:

1. $\sin^2 x + \cos^2 x = 1.$
2. $\sin 2\,x = 2 \sin x \cos x.$
3. $\cos 2\,x = \cos^2 x - \sin^2 x.$
4. $\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x.$

262. The Hyperbolic Functions. — In the expansions for $\sin x$ and $\cos x$ given at the beginning of (**261**) replace x by ix and *define* the results to be $\sin ix$ and $\cos ix$ respectively. We obtain

$$\sin ix = i\left(x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \cdots\right);$$

$$\cos ix = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \cdots.$$

These equations we consider as defining the sine and cosine of the imaginary quantity ix.

Multiply the first equation by i and subtract the result from the second. We obtain

$$\cos ix - i \sin ix = e^x.$$

Change x to $-x$;

$$\cos ix + i \sin ix = e^{-x}.$$

(Note that $\sin ix = -\sin(-ix)$ by the definition of $\sin ix$.)

Combining the last two equations by addition and subtraction, we find

$$\cos ix = \frac{e^x + e^{-x}}{2}; \quad \sin ix = i\,\frac{e^x - e^{-x}}{2}.$$

We now define

Hyperbolic cosine of x ($= \cosh x$) $= \cos ix$;

Hyperbolic sine of x ($= \sinh x$) $= \frac{1}{i} \sin ix$.

Then

$$\cosh x = \frac{e^x + e^{-x}}{2}; \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

These functions are related to the hyperbola somewhat as the circular functions to the circle.

The remaining hyperbolic functions are defined by the equations

$$\tanh x = \frac{\sinh x}{\cosh x}; \quad \coth x = \frac{1}{\tanh x};$$

$$\operatorname{sech} x = \frac{1}{\cosh x}; \quad \operatorname{csch} x = \frac{1}{\sinh x}.$$

Exercises. Show that:

1. $\sinh 0 = 0; \quad \cosh 0 = 1.$
2. $\sinh \pi i = 0; \quad \cosh \pi i = -1.$
3. $\sinh \frac{\pi i}{2} = i; \quad \cosh \frac{\pi i}{2} = 0.$
4. $\sinh(-x) = -\sinh x.$
5. $\cosh(-x) = \cosh x.$
6. $\cosh^2 x - \sinh^2 x = 1.$
7. $\operatorname{sech}^2 x = 1 - \tanh^2 x.$
8. $-\operatorname{csch}^2 x = 1 - \coth^2 x.$

Draw the graphs of the equations (see tables):

9. $y = e^x.$
10. $y = e^{-x}.$
11. $y = \cosh x.$
12. $y = \sinh x.$

CHAPTER XVIII

Permutations. Combinations. Chance

263. Permutations. — A *permutation* is a definite order or arrangement of a group of objects, or of part of the group.

Let there be a group of n distinct objects. The number of possible arrangements, taking r of these objects at a time is called the *number of permutations of n things r at a time,* and is denoted by ${}_nP_r$.

Theorem 1. *The number of permutations of n things r at a time is*

$$ {}_n\boldsymbol{P}_r = \boldsymbol{n}\,(\boldsymbol{n}-\boldsymbol{1})\ \ldots\ (\boldsymbol{n}-\boldsymbol{r}+\boldsymbol{1}). $$

Proof. Evidently $\quad {}_nP_1 = n.$

Now with each of the n objects we may pair any one of the remaining $n - 1$ objects.

Hence $\quad {}_nP_2 = n\,(n-1).$

With each one of these $n\,(n-1)$ permutations containing 2 objects we may associate one of the remaining $n - 2$ objects.

Hence $\quad {}_nP_3 = n\,(n-1)\,(n-2).$

Proceeding in this way we obtain the formula stated.

When $\quad r = n$ we have

$$ {}_nP_n = \underline{|n}. $$

Exercises.

1. How many numbers of four figures each can be formed from the digits 1, 2, 3, 4?

2. How many 3-figure numbers can be formed from the digits 1, 2, 3, 4, 5?

3. How many numbers greater than 1000 can be formed from the digits 1, 3, 5, 7, 9?

4. How many changes can be rung with 8 bells, 4 at a time?

264. Combinations. — A combination is a group of objects, without reference to their arrangement.

The number of different groups or combinations of n objects, each group containing r objects, is called the number of combinations of n things r at a time, and is denoted by ${}_nC_r$.

Theorem 2. The number of combinations of n things r at a time is

$$ {}_nC_r = \frac{{}_nP_r}{\lfloor\underline{r}} = \frac{n(n-1) \cdots (n-r+1)}{\lfloor\underline{r}}. $$

Proof. Suppose all the combinations of the n things r at a time to be written down. Each group so written will yield, by permuting its objects in all possible ways, $\lfloor\underline{r}$ permutations. Hence there are $\lfloor\underline{r}$ times as many permutations as combinations, or

$$ \lfloor\underline{r}\, {}_nC_r = {}_nP_r = n(n-1) \ldots (n-r+1). $$

Hence the theorem.

Exercises.

1. How many triangles can be formed from 6 points, no three points being collinear?

2. How many tetrahedrons can be formed from 12 points, no four points being coplanar?

3. How many committees of 3 persons each can be formed from a club of 10 persons?

4. Show that ${}_nC_r = {}_nC_{n-r}$.

(This is a convenient formula when r is nearly as large as n. It is then shorter to calculate ${}_nC_{n-r}$.)

5. Show that ${}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n$.

(Expand $(1+x)^n$ and put $x = 1$; ${}_nC_0$ is *defined* to be 1.)

6. How many committees, consisting of from 1 to 9 members, can be formed from a club of 10 persons?

7. Find the value of ${}_{20}C_{18}$.

265. Theorem 3. — *The number of permutations of n things n at a time, when p things are alike, is*

$$ P = \frac{\lfloor\underline{n}}{\lfloor\underline{p}}. $$

Proof. Let P be the number of permutations sought, and suppose them written down. If now the p things in question were unlike, by permutating them among themselves each of the P permutations would yield $\lfloor\underline{p}$ permutations; the total number of permutations so formed would be $\lfloor\underline{p}\, P$ and must equal ${}_nP_n$ or $\lfloor\underline{n}$. Hence the theorem.

Similarly, the number of permutations of n things n at a time, when p things are all of one kind, and q of a second kind, will be

$$P = \frac{\lfloor n}{\lfloor p \lfloor q},$$

and so on.

266. Exercises.

1. How many permutations of seven letters each can be formed from the letters of the word "arrange"?
2. How many permutations of 11 letters each can be formed from the letters of the word "Mississippi"?
3. How many words, each containing a vowel and two consonants, can be formed from 4 vowels and 6 consonants?
4. How many even numbers of four figures each can be formed from the digits 1, 2, 3, 4, 5, 6?
5. How many elevens can be chosen from 20 players if only 6 of the 20 are qualified to play behind the line?
6. As in 5, if in addition, only 2 men are qualified for center.
7. How many sums can be formed with one coin of each denomination, from a cent to a dollar?
8. As in 7, except that there are two coins of each denomination.
9. If two coins are tossed, in how many ways may they fall?
10. As in 9, for 10 coins.
11. If two dice are thrown, in how many ways may they turn up?
12. As in 11, for 3 dice.

267. Probability or Chance. — If a bag contains 4 white and 3 black balls, and a ball is drawn at random, what is the chance that it be white?

In order to solve this problem we first define chance or probability.

Definition. The measure of the probability of the occurrence of an event is taken to be the quotient,

$$\frac{\text{number of favorable ways}}{\text{total number of possible ways}}.$$

In the problem above, since there are 7 balls altogether, there are 7 possible ways of drawing one ball; of these 4 are favorable, since there are 4 white balls. Hence the chance that a white ball be drawn is $\frac{4}{7}$.

Similarly the chance for a black ball is $\frac{3}{7}$.

If an event can happen in a ways, and fail in b ways, then, by the definition, the chance that it will happen is $\frac{a}{a+b}$, and that it will fail is $\frac{b}{a+b}$.

Since the event must either happen or fail, the probability for which is $\frac{a}{a+b}+\frac{b}{a+b}=1$, we have 1 as the mathematical symbol for certainty.

If p is the probability that an event will happen, $1-p$ is the probability that it will fail.

Example 1. From a bag containing 4 white and 3 black balls, 2 balls are drawn at random.

(*a*) What is the chance that both be white?

Number of favorable ways: ${}_4C_2 = 6$.

Number of possible ways: ${}_7C_2 = 21$.

Hence the required chance is: $p = \frac{6}{21} = \frac{2}{7}$.

(*b*) What is the chance that at least one be white?

Favorable cases: both white, ${}_4C_2 = 6$;

one white, other black, $3 \times 4 = 12$.

∴ Total number of favorable cases is 18.

Number of possible cases, as before, 21.

Hence $p = \frac{18}{21} = \frac{6}{7}$.

A shorter method is as follows: The probability that both balls be black is $\frac{{}_3C_2}{{}_7C_2} = \frac{3}{21} = \frac{1}{7}$. Hence the chance that at least one be white is $1 - \frac{1}{7} = \frac{6}{7}$.

Example 2. From 12 tickets, numbered 1, 2, . . . 12, four are drawn at random.

(*a*) What is the probability that they bear even numbers?

Since 6 tickets bear even numbers, the number of favorable cases is ${}_6C_4$. The total number of ways of drawing 4 tickets from 12 is ${}_{12}C_4$. Hence

$$p = \frac{{}_6C_4}{{}_{12}C_4} = \frac{6 \cdot 5 \cdot 4 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9} = \frac{1}{33}.$$

(*b*) What is the chance that two bear even, the other two odd numbers?

We can select two tickets bearing even numbers in ${}_6C_2$ ways; also two bearing odd numbers in ${}_6C_2$ ways. Combining any one of the first with any one of the second gives ${}_6C_2 \times {}_6C_2$ favorable ways. Hence

$$p = \frac{{}_6C_2 \times {}_6C_2}{{}_{12}C_4} = \frac{5}{11}.$$

268. Exercises.

1. If 5 coins are tossed, what is the chance of three heads?
2. If 5 coins are tossed, what is the chance of at least two heads?
3. If 3 balls are drawn from a bag containing 5 white and 4 black balls, what is the chance that all three are white?
4. In exercise 3, what is the chance of drawing 2 white balls and one black ball?
5. In exercise 3, what is the chance of drawing at least one white ball?
6. What is the chance of two sixes in a single throw of two dice?
7. What is the chance of throwing three sixes in a single throw with three dice?
8. Three dice are thrown. What is the chance that the sum of the numbers turned up is 11?
9. As in 8, except that the sum is to be 7.
10. Six cards are drawn from a pack of 52. What is the chance of three aces?
11. Six cards are drawn from a pack of 52. What is the chance that all are of the same suite?

269. Compound Probabilities.

Definition. Two events are said to be independent when the occurrence of one does not affect that of the other.

Theorem 4. *The chance that both of two independent events shall happen is the product of their separate probabilities.*

Proof. Suppose the first event happens in a ways and fails in b ways, out of a $a + b$ possible ways, and that the second happens in a' ways and fails in b' ways, out of a total of $a' + b'$ ways.

Combining each of the a favorable ways of the first event with each of the a' favorable ways of the second, we have aa' favorable cases. The total number of possible cases is $(a + b)(a' + b')$. Hence

$$p = \frac{aa'}{(a+b)(a'+b')} = \frac{a}{a+b} \times \frac{a'}{a'+b'}$$

which is the product of the separate probabilities of the two events.

As an immediate extension, we have the

Theorem 5. *If the probabilities of several independent events be $p_1, p_2, \ldots p_n$, the probability that all will happen is*

$$P = p_1 \times p_2 \times \cdots \times p_n.$$

Example. From a bag containing 4 white and 3 black balls, 2 balls are drawn in succession. What is the chance that both are white?

On the first drawing the chance for a white ball is $\frac{4}{7}$; on the second, $\frac{3}{6}$. The probability of both events is therefore

$$\frac{4}{7}\times\frac{3}{6}=\frac{2}{7}.$$

Definition. Two events are said to be dependent when the occurrence of one of them affects that of the other.

Theorem 6. *Of n dependent events, let the chance that the first will happen be p_1, the chance that the second then follows be p_2, that the third then follows be p_3, and so on. The chance that all these events shall happen is then*

$$P = p_1 \times p_2 \times p_3 \times \cdot \cdot \cdot \, p_n$$

This is an immediate consequence of the preceding theorem.

Theorem 7. *If p be the chance that an event will happen in one trial, the chance that it will happen just r times in n trials is*

$$\boldsymbol{P = {}_nC_r\, p^r(1-p)^{n-r}}.$$

Proof. The chance that r trials out of n shall succeed is p^r, and that the other $n-r$ trials shall fail is $(1-p)^{n-r}$. Hence the probability of success in r particular trials and of failure in the $n-r$ other trials is $p^r(1-p)^{n-r}$. But of the n trials, any r may be the successful ones, which gives ${}_nC_r$ possibilities, each having a probability $p^r(1-p)^{n-r}$. Hence the result stated.

Examples.

1. In a class of 3 students, A solves on the average 9 problems out of 10, B 8 out of 10, C 7 out of 10. What is the chance that a problem, presented to the class, will be solved?

The problem will be solved unless all three students fail, the probability for which is

$$\frac{1}{10}\times\frac{2}{10}\times\frac{3}{10}=\frac{3}{500}.$$

Hence the chance that the problem will be solved is

$$1-\frac{3}{500}=\frac{497}{500}.$$

2. Two bags each contain 5 black balls, and a third bag contains 5 black and 5 white balls. What is the chance of drawing a white ball from one of the bags selected at random?

The chance that the bag containing white balls be chosen is $\frac{1}{3}$. The chance that a white ball be now drawn from this bag is $\frac{1}{2}$. Hence the probability that both events happen and that a white ball be drawn is

$$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}.$$

3. A coin is tossed 10 times. What is the chance for just 3 heads?

The probability of a head in one trial is $\frac{1}{2}$. Hence

$$_nC_r p^r (1 - p)^{n-r} = {}_{10}C_3\left(\frac{1}{2}\right)^3\left(1 - \frac{1}{2}\right)^7 = \frac{15}{128}.$$

270. Exercises.

1. Three hats each contain 5 tickets, those in two of the hats being numbered 1, 2, . . . 5, and those in the third hat being blank. What is the chance of drawing a ticket bearing an even number from one of the hats selected at random?

2. If in exercise 1 two tickets be drawn from a hat chosen at random, what is the chance that both bear even numbers?

3. If each of two persons draw a ticket from one of the hats in exercise 1, the first ticket being replaced before the second is drawn, what is the chance that both persons draw the same number? What is the chance that both draw blanks?

4. If a coin be tossed 10 times, what is the chance for at least 7 heads?

5. How many different sets of throws can be made with a coin, each set consisting of 5 successive throws?

6. The chance that a person aged 25 years will live to be 75 is $\frac{7}{24}$. What is the chance that, of three couples married at the age of 25, at least one shall live to celebrate their golden wedding?

7. A bag contains 10 white, 6 black, and 4 red balls. Find the chance that, of three balls drawn, there shall be one of each color.

8. A gunner hits the target on an average 7 times out of 10. What is the chance that 5 consecutive shots shall hit the target?

9. Two dice are thrown. Find the chance that the sum of the numbers turned up shall be even.

CHAPTER XIX

THEORY OF EQUATIONS

271. We shall refer to the equation

$$(1)\qquad p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$$

as the **standard form of the equation of nth degree;** p_0x^n is called the **leading term** and p_n the **constant** (or **absolute**) **term.**

The coefficient of the leading term may be made equal to unity by dividing the whole equation by this coefficient.

When all the terms written in equation (1) are present, the equation is said to be **complete;** when one or more terms are absent, the equation is said to be **incomplete.** An incomplete equation may be made complete by supplying the missing terms with zero coefficients.

We shall represent the polynomial forming the left member of equation (1) by $f(x)$; $f(a)$ shall denote the value of this polynomial when $x = a$, $f(b)$ the value when $x = b$, and so on.

A **root** of an equation is a value of x which satisfies the equation; hence a is a root of the equation $f(x) = 0$ if $f(a) = 0$.

In the present chapter we shall consider methods of finding the roots of the equation $f(x) = 0$.

272. Factor Theorem. — *If a is a root of the equation $f(x) = 0$, then $f(x)$ is divisible by $(x - a)$, and conversely.*

Proof. Divide $f(x)$ by $(x - a)$; let Q be the quotient, R the remainder. Then

$$f(x) = (x - a)Q + R.$$

Putting $x = a$, we obtain $R = 0$, since $f(a) = 0$ by hypothesis. Hence $f(x)$ is divisible by $(x - a)$ without a remainder.

Conversely, assume

$$f(x) = (x - a)Q.$$

Put $x = a$ and we have $f(a) = 0$; hence a is a root of $f(x) = 0$. [See also (**11**), (f).]

273. Depressed Equation. — When a is a root of the equation $f(x) = 0$, we may write

$$f(x) = (x - a)\,Q.$$

Any other value of x which reduces $f(x)$ to zero must also reduce Q to zero, and is therefore a root of the equation $Q = 0$.

But if $f(x)$ is of degree n, Q will be of degree $n - 1$. Hence when one root of an equation is known, the other roots may be found by solving an equation of degree one less than that of the given equation, and whose left member is found by dividing the left member of the given equation by (x − *the root*).

The new equation is called the *depressed equation*.

Exercises. Show that each of the following equations has the root indicated, and find the other roots:

1. $x^3 - 9x^2 + 26x - 24 = 0$; $x = 2$.
2. $x^4 + 3x^2 - 8x - 24 = 0$; $x = -3$.
3. $3x^3 - 14x^2 + 17x - 6 = 0$; $x = \frac{2}{3}$.

274. Number of Roots. — We *assume* that every equation of the form (1), (**271**), has at least one root, that is, there exists at least one value of x, real or imaginary, which satisfies the equation. It can then be shown that *an equation of the nth degree has just n roots.*

For, let a_1 be a root. Form the depressed equation by removing from $f(x)$ the factor $x - a_1$, and let the new equation, of degree $n - 1$, be $f_1(x) = 0$. By the above assumption, this equation has a root, say a_2. Removing from $f_1(x)$ the factor $x - a_2$, we obtain a new equation, $f_2(x) = 0$, of degree $n - 2$, and so on. After $n - 1$ steps, by which $n - 1$ roots are removed, we have an equation of the first degree which gives one more root. Hence there are just n roots.

275. To Form an Equation Having Given Roots. — Let it be required to form an equation whose roots are $a_1, a_2, a_3, \ldots a_n$.

Obviously the required equation is

$$A\,(x - a_1)(x - a_2)(x - a_3)\;\ldots\;(x - a_n) = 0,$$

A being an arbitrary constant.

Exercises. Form the equations whose roots are:

1. 1, 2, 3.
2. 1, −1, 2.
3. 2, 2, −2, 0.
4. −1, −2, −3, −4.
5. $\pm 1, \frac{1}{2}, \frac{1}{3}$.
6. $-\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

(Write the results from exercises 5 and 6 with integral coefficients.)

276. Relations between Coefficients and Roots. — In the case of 2, 3, and 4 roots respectively we find on expanding,

$$(x - a_1)(x - a_2) = x^2 - (a_1 + a_2)\,x + a_1a_2.$$

$$(x - a_1)(x - a_2)(x - a_3) = x^3 - (a_1 + a_2 + a_3)\,x^2 + (a_1a_2 + a_2a_3 + a_1a_3)\,x - a_1a_2a_3.$$

$$(x - a_1)(x - a_2)(x - a_3)(x - a_4) = x^4 - (a_1 + a_2 + a_3 + a_4)x^3 + (\cdots)\,x^2 - (\cdots)\,x + a_1a_2a_3a_4.$$

We here observe three facts, namely:

1. *The coefficient of the leading term is unity;*
2. *The coefficient of the second term is the negative sum of the roots;*
3. *The constant term is the product of the roots, plus when the number of roots is even, minus when odd.*

We shall show by induction that these results are true in general. Assume them to be true for $n - 1$ roots; then if the equation whose roots are $a_1, a_2, \ldots a_{n-1}$, be

$$x^{n-1} + q_1x^{n-2} + \cdots + q_{n-1} = 0,$$

we have by hypothesis,

$$q_1 = -(a_1 + a_2 + \cdots + a_{n-1}); \quad q_{n-1} = (-1)^{n-1}a_1a_2 \ldots a_n.$$

Multiplying the above equation by $(x - a_n)$, which introduces the new root a_n, we find on collecting in powers of x:

$$x^n + (q_1 - a_n)\,x^{n-1} + \cdots - a_nq_{n-1} = 0,$$

or,

$$x^n - (a_1 + a_2 + \cdots + a_{n-1} + a_n)\,x^{n-1} + \cdots + (-1)^n a_1a_2 \ldots a_{n-1}a_n = 0.$$

Hence if the results are true for the case of $n - 1$ roots, they hold also for n roots. But they are true for 4 roots, hence also for 5, hence for 6, and so on.

Exercise. Show by induction that the coefficient of the third highest power of x equals the sum of the products of the roots taken two at a time.

277. Fractional Roots. — *An equation with integral coefficients, in which the coefficient of the leading term is unity, cannot have as a root a rational fraction in its lowest terms.*

Proof. Assume that the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$$

has integral coefficients and that one of its roots is $\frac{a}{b}$, where a and b are integers, relatively prime. Putting $x = \frac{a}{b}$ we have,

$$\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + p_2\left(\frac{a}{b}\right)^{n-2} + \cdots + p_{n-1}\left(\frac{a}{b}\right) + p_n = 0.$$

Multiplying through by b^{n-1} and transposing,

$$\frac{a^n}{b} = -(p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_{n-1}ab^{n-2} + p_nb^{n-1}).$$

Here we have a fraction in its lowest terms equal to an integer, which is impossible. Hence $\frac{a}{b}$ cannot be a root.

Corollary. Any rational root of an equation whose coefficients are integers and whose leading coefficient is unity must be an integer.

278. Imaginary Roots. — *If the general equation of nth degree, with real coefficients, has an imaginary root $a + ib$, then also the conjugate imaginary, $a - ib$, is a root.*

Proof. Assume that $a + ib$ is a root of the equation

$$f(x) \equiv x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

Then

$$(a + ib)^n + p_1(a + ib)^{n-1} + p_2(a + ib)^{n-2} + \cdots + p_{n-1}(a + ib) + p_n = 0.$$

Expanding the binomials, reducing all powers of i to ± 1 or $\pm i$, and collecting terms, we have a result of the form

$$f(a + ib) = P + iQ = 0.$$

Hence $P = 0$ and $Q = 0$. (**258.**)

Now substitute $a - ib$ for x and proceed as before. The result will be

$$f(a - ib) = P - iQ,$$

since the only difference is in the sign of i. But $P = 0$ and $Q = 0$, hence $P - iQ = 0$, or $f(a - ib) = 0$. Therefore $a - ib$ is a root.

We may state our result as follows: *Imaginary roots, if present at all, always occur in conjugate pairs.*

279. Multiple Roots. — *When an equation has two or more roots equal to the same value "a," then "a" is called a multiple root.*

Suppose that the equation

$$f(x) = 0$$

has m roots, each equal to a. Then

$$f(x) = (x - a)^m Q,$$

where Q is a new polynomial.

Let $f'(x)$ denote the first derivative of $f(x)$ with respect to x; then

$$f'(x) = (x - a)^m \frac{dQ}{dx} + m(x - a)^{m-1} Q.$$

This shows that $f'(x)$ contains the factor $(x - a)^{m-1}$, and hence that, *if $f(x) = 0$ contains a root "a" repeated m times, $f'(x) = 0$ will contain this root repeated $m - 1$ times; $f(x)$ and $f'(x)$ will then have the factor $(x - a)^{m-1}$ in common.*

Hence we have the following rule for finding multiple roots of the equation $f(x) = 0$.

Find the H. C. F. (**13**) *of $f(x)$ and $f'(x)$; to a factor $(x - a)^{m-1}$ of the H. C. F. corresponds a factor $(x - a)^m$ of $f(x)$.*

280. Exercises. Test for multiple roots and find all the roots of the equations:

1. $x^3 - 3x^2 + 4 = 0.$

2. $x^3 - 3x - 2 = 0.$

3. $x^4 - 2x^3 - 11x^2 + 12x + 36 = 0.$

4. $x^4 - 2x^3 - 39x^2 + 40x + 400 = 0.$

5. $x^4 - 3x^3 - 7x^2 + 15x + 18 = 0.$

6. $x^4 + 4x^3 - 16x - 16 = 0.$

7. $x^4 - x^3 - 3x^2 + 5x - 2 = 0.$

8. $4x^4 + 6x^3 + 5x^2 - 6x + 4 = 0.$

9. $9x^4 - 54x^3 + 80x^2 + 6x - 9 = 0.$

10. $72x^5 - 276x^4 + 278x^3 + 45x^2 - 108x - 27 = 0.$

281. Transformation of Equations. — In the following discussion we assume that any missing powers of x are inserted, supplied with zero coefficients, so as to make the equation formally complete. We consider the equation

$$f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

I. To change the signs of the roots.

Put $x = -y$. We obtain,

$$p_0(-y)^n + p_1(-y)^{n-1} + p_2(-y)^{n-2} + \cdots + p_{n-1}(-y) + p_n = 0,$$

or, on multiplying through by $(-1)^n$,

$$p_0y^n - p_1y^{n-1} + p_2y^{n-2} - \cdots + (-1)^{n-1}p_{n-1}y + (-1)^np_n = 0.$$

Hence, *to change the signs of the roots, change the signs of alternate coefficients, beginning with the second term.*

II. To multiply the roots by a constant factor, *m*.

Replace x by $\frac{y}{m}$ (so that $y = mx$).

Then

$$p_0\left(\frac{y}{m}\right)^n + p_1\left(\frac{y}{m}\right)^{n-1} + p_2\left(\frac{y}{m}\right)^{n-2} + \cdots + p_{n-1}\left(\frac{y}{m}\right) + p_n = 0.$$

Multiplying through by m^n, we have,

$$p_0y^n + mp_1y^{n-1} + m^2p_2y^{n-2} + \cdots + m^{n-1}p_{n-1}y + m^np_n = 0.$$

Hence, *to multiply the roots by a constant factor m, multiply the coefficients in order, beginning with the second, by $m, m^2, m^3, \ldots m^n$.*

When $m = -1$ we obtain the preceding rule for changing the signs of the roots.

III. To increase the roots by a constant quantity, *h*.

Replace x by $y - h$ (so that $y = x + h$). Then

$$p_0(y-h)^n + p_1(y-h)^{n-1} + p_2(y-h)^{n-2} + \cdots + p_{n-1}(y-h) + p_n = 0.$$

Expanding the binomials and collecting in powers of y, we obtain a result of the form,

$$p_0y^n + P_1y^{n-1} + P_2y^{n-2} + \cdots + P_{n-1}y + P_n = 0.$$

We shall now show how to obtain the coefficients $P_1, P_2, \ldots P_n$.

Replacing y in the last equation by $x + h$, the result must be the original equation, $f(x) = 0$. Hence

$$f(x) = p_0(x+h)^n + P_1(x+h)^{n-1} + P_2(x+h)^{n-2} + \cdots + P_{n-1}(x+h) + P_n.$$

This shows that if $f(x)$ be divided by $x + h$, the remainder is P_n. If the quotient be divided by $x + h$, the remainder is P_{n-1}; dividing the second quotient by $(x + h)$, the remainder is P_{n-2}, and so on.

Hence, *to increase the roots of the equation by h, divide $f(x)$ by $x+h$, then divide the quotient by $x+h$, divide the new quotient by $x+h$, and so on. The successive remainders are, in order,*

$$P_n, P_{n-1}, P_{n-2}, \ldots P_1.$$

A concise method for performing the required divisions will be explained in the next article.

282. Synthetic Division. — When h and the coefficients p_0, p_1, p_2, . . . p_n are integers, the work of dividing $f(x)$ may be performed by the method of *synthetic division.* We shall illustrate this by increasing the roots of the equation

$$x^3 - 8x - 15 = 0$$

by 2.

Performing the first division at length, we have:

$$\begin{array}{rrrr|l} x^3 + 0x^2 & - 8x & - 15 & & x + 2 \\ \underline{x^3 + 2x^2} & & & & \overline{x^2 - 2x - 4} \text{ quotient.} \\ -2x^2 & - 8x & & & \\ \underline{-2x^2} & \underline{- 4x} & & & \\ & -4x & - 15 & & \\ & \underline{-4x} & \underline{- 8} & & \\ & & - 7 & \text{remainder.} & \end{array}$$

We first shorten this operation by omitting to write the powers of x, using only the *detached coefficients,* thus:

$$\begin{array}{rrrr|l} 1 + 0 & - 8 & - 15 & & 1 + 2 \\ \underline{1 + 2} & & & & \overline{1 - 2 - 4} \\ -2 & - 8 & & & \\ \underline{-2} & \underline{- 4} & & & \\ & -4 & - 15 & & \\ & \underline{-4} & \underline{- 8} & & \\ & & - 7 & & \end{array}$$

This may be written more compactly as follows:

$$\begin{array}{lrrrr|l} & 1 + 0 & - 8 & - 15 & & \underline{+2} \\ & + 2 & - 4 & - 8 & & \\ \text{1st quotient} & \overline{1 - 2} & - 4 & | - 7 & \text{remainder.} & \end{array}$$

Dividing the quotient by $x+2$ we have,

$$\begin{array}{l|l} 1-2-4 & +2 \\ \quad +2-8 & \\ \hline 1-4 & +4 \text{ remainder.} \end{array}$$

2nd quotient

Dividing the second quotient by $x+2$ we have,

$$\begin{array}{l|l} 1-4 & +2 \\ \quad +2 & \\ \hline 1 & -6 \text{ remainder.} \end{array}$$

3rd quotient

The whole operation may now be written thus:

$$\begin{array}{l|l} 1+0-8-15 & +2 \\ \quad +2-4-\ \ 8 & \\ \hline 1-2-4 & -\ \ 7 \text{ 1st remainder} \\ \quad +2-8 & \\ \hline 1-4 & +4 \text{ 2nd remainder} \\ \quad +2 & \\ \hline 1 & -6 \text{ 3rd remainder.} \end{array}$$

Then the transformed equation is:

$$x^3-6x^2+4x-7=0.$$

To diminish the roots of an equation by h, proceed as above with $x-h$ in place of $x+h$. As an example, we diminish by 4 the roots of the equation

$$x^4-5x^3+7x^2-17x+11=0.$$

$$\begin{array}{l|l} 1-\ \ 5+\ \ 7-17+11 & -4 \\ \quad -\ \ 4+\ \ 4-12+20 & \\ \hline 1-\ \ 1+\ \ 3-\ \ 5 & -\ \ 9 \text{ 1st remainder} \\ \quad -\ \ 4-12-60 & \\ \hline 1+\ \ 3+15 & +55 \text{ 2nd remainder} \\ \quad -\ \ 4-28 & \\ \hline 1+\ \ 7 & +43 \text{ 3rd remainder} \\ \quad -\ \ 4 & \\ \hline 1 & +11 \text{ 4th remainder.} \end{array}$$

Hence the transformed equation is:

$$x^4+11x^3+43x^2+55x-9=0.$$

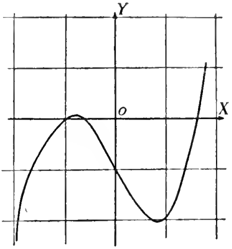
In using the method of synthetic division note that the coefficient of the leading term remains unchanged.

283. The graph of the equation $y = f(x)$, when

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdot \cdot \cdot + p_{n-1}x + p_n.$$

To construct the graph which shall represent the fluctuating values of y as x varies, we assume a series of numerical values for x, calculate the corresponding values of y, and plot the points (x, y). On drawing a smooth curve through these points, we obtain a graph such as that in the figure, which represents the equation

$$y = x^3 - 2x - 1.$$



$y = x^3 - 2x - 1.$

Here a set of corresponding values of x and y are:

$$x = \quad 0, \quad 1, 2, \ldots, -1, -2, \ldots;$$
$$y = -1, -2, 3, \ldots, \quad 0, -5, \ldots.$$

Since the curve crosses the x-axis when $y = 0$, we see that *the abscissas of the points where the graph of the equation $y = f(x)$ crosses the x-axis (called the x-intercepts of the graph) are the real roots of the equation $f(x) = 0$.*

An inspection of the above graph shows that one root of the equation $x^3 - 2x - 1 = 0$ is -1, another root lies between -1 and 0, and the third between $+1$ and $+2$. On removing the factor $x + 1$ from this equation, the depressed equation is $x^2 - x - 1 = 0$. Hence the exact values of the other two roots are $\frac{1}{2}(1 \pm \sqrt{5})$, or approximately, $+1.62$ and -0.62.

284. Effect of Changing the Constant Term. — Suppose that we add a quantity k to the constant term of $f(x)$, so that the equation

$$y = f(x)$$

becomes

$$y = f(x) + k.$$

Suppose the curve $y = f(x)$ to be plotted; on adding k to each of its ordinates, we obtain the graph of $y = f(x) + k$. That is, *if* *k be added to the constant term of the equation $y = f(x)$, the graph is displaced vertically through the distance k, upward if k is plus, downward if k is minus.*

As an example, consider the equations

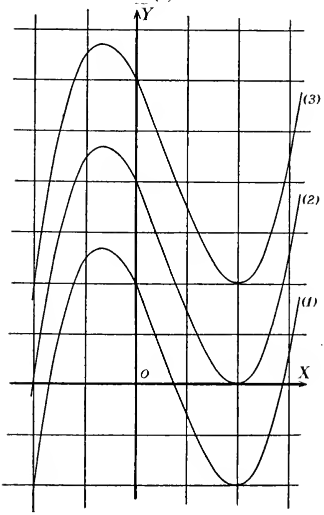
(1) $$y = \tfrac{1}{2}x^3 - x^2 - 2x + 2,$$

(2) $$y = \tfrac{1}{2}x^3 - x^2 - 2x + 4,$$

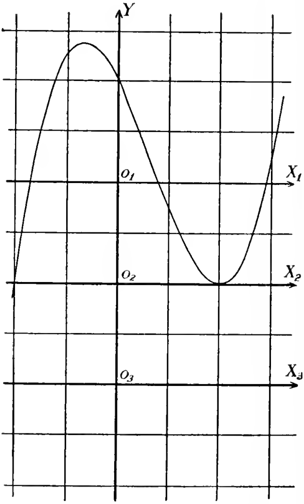
(3) $$y = \tfrac{1}{2}x^3 - x^2 - 2x + 6.$$

The graphs are shown in figure (*a*). The curves are of precisely the same form, but (2) lies two units higher than (1), and (3) two units higher than (2).

(*a*)



(*b*)



Instead of displacing the curve vertically, say upward, the same effect is produced in the graph by moving the x-axis an equal distance downward. Thus equations (1), (2), (3) are represented graphically by the curve in figure (*b*), y being counted from the lines O_1X_1, O_2X_2, O_3X_3 respectively.

285. Occurrence of Imaginary Roots in Pairs. — We can now consider article (**277**) geometrically. Thus in the first figure of (**284**), graph (1) shows that the equation

$$\tfrac{1}{2}x^3 - x^2 - 2x + 2 = 0$$

has three real unequal roots; replacing 2 by 4, the two positive roots become equal; that is, the equation

$$\tfrac{1}{2}x^3 - x^2 - 2x + 4 = 0$$

has three real roots, two of which are equal; finally on replacing 4 by 6, the two equal roots become imaginary; that is, the equation

$$\tfrac{1}{2}x^3 - x^2 - 2x + 6 = 0$$

has one real root and two imaginary roots.

In general, by changing the constant term in $f(x)$, the graph of $y = f(x)$ may be raised or lowered so that one of the "elbows" of the curve, which at first is cut by the x-axis, will become tangent to the x-axis, and on further changing the constant the x-axis will fail to intersect this elbow. Thus two real unequal roots first become equal, then imaginary.

286. Exercises. Multiply the roots of the equation

1. $x^3 + x^2 - x - 1 = 0$ by 2;

2. $x^3 - 2x + 1 = 0$ by -2;

3. $x^3 - 48x - 112 = 0$ by $\tfrac{1}{2}$;

4. $x^4 + 6x^3 + 3x^2 - 26x - 24 = 0$ by $-\tfrac{1}{2}$.

Multiply the roots of the following equations by the smallest factor which will make all coefficients integers

5. $x^2 + x + \tfrac{1}{4} = 0.$

6. $\tfrac{1}{2}x^3 - x^2 + \tfrac{1}{32} = 0.$

7. $x^2 - \tfrac{1}{2}x - \tfrac{1}{8} = 0.$

8. $x^3 - .1x^2 + .01x = 0.$

9. $x^3 + \tfrac{1}{9}x^2 - \tfrac{1}{81} = 0.$

10. $x^4 + 1.2x^2 - .225x + .015 = 0.$

Increase the roots of the equation

11. $x^3 - 3x^2 + 4 = 0$ by 2.

12. $4x^3 - 3x - 1 = 0$ by 3.

13. $x^4 - 2x^3 - 11x^2 + 12x + 36 = 0$ by -2.

14. $x^4 - 2x^3 - 39x^2 + 40x + 400 = 0$ by -4.

In the following equations increase the roots by a quantity such that the term involving the second highest power of x shall disappear.

15. $x^3 - 3x^2 + 2 = 0.$

16. $x^3 - 2x^2 + 1 = 0.$

17. $x^3 - 3x^2 - 6x + 1 = 0.$

18. $x^4 - 4x^3 - 8x + 32 = 0.$

In the following equations change the constant so that two roots shall become imaginary.

19. $x^3 - x^2 - 2x = 0.$

20. $x^3 - 3x^2 + 3 = 0.$

21. $x^3 - 3x - 2 = 0.$

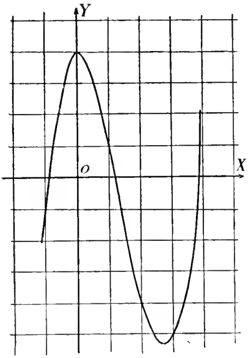
22. $x^3 - x^2 - x + 1 = 0.$

Solve the following equations, given one root.

23. $x^3 - 2x^2 + x - 2 = 0; \quad x = \sqrt{-1}.$

24. $2x^4 - 3x^3 + 5x^2 - 6x + 2 = 0; \quad x = -2\sqrt{-1}.$

25. $x^5 - 8x^3 - 8x^2 + 64 = 0; \quad x = -1 - \sqrt{-3}.$



287. Approximation to the Roots of an Equation. — In this article we shall illustrate a method for obtaining, to any desired degree of accuracy, any real root of an algebraic equation. As an example we shall obtain, to four decimal places inclusive, one of the roots of the equation

$$(1) \quad f(x) \equiv x^3 - 4x^2 + 4 = 0.$$

The graph is given in the figure.

First Step. Location of Real Roots. We first locate the real roots by trial. As a set of corresponding values of x and $f(x)$ we have

$$x = -2, \quad -1, \quad 0, \quad +1, \quad +2, \quad +3, \quad +4.$$
$$f(x) = -20, \quad -1, \quad +4, \quad +1, \quad -4, \quad -5, \quad +4.$$

When $f(x)$ changes sign, the graph crosses the x-axis, and at least one root must lie between the corresponding values of x. Hence there is a root between -1 and 0, another between $+1$ and $+2$, and a third between $+3$ and $+4$. But there cannot be more than three roots, since a cubic expression cannot contain more than three linear factors. Hence there is just one root in each of the above intervals.

We shall proceed to obtain the root between 1 and 2.

Second Step. Diminish the roots of the given equation by the integral part of the root required (**281**).

$$\begin{array}{l|l}
1 - 4 + 0 + 4 & \mathbf{-1} \\
\quad - 1 + 3 + 3 & \\
\hline
1 - 3 - 3 & \mathbf{+1} \\
\quad - 1 + 2 & \\
\hline
1 - 2 & \mathbf{-5} \\
\quad - 1 & \\
\hline
1 & \mathbf{-1}
\end{array}$$

The transformed equation is

(2) $$x^3 - x^2 - 5x + 1 = 0.$$

Since (1) has a root between 1 and 2, (2) must have a root between 0 and 1, that is, a decimal root. To make this root an integer, we take the

Third Step. Multiply the roots of the transformed equation by 10 (**281**).

The new equation is

(3) $$x^3 - 10x^2 - 500x + 1000 = 0.$$

The root of (3) between 0 and 10 will give the first decimal of the required root of (1). If we neglect the terms in x^3 and x^2 in (3) we get an approximate value, $x = 2$. Putting $x = 2$ in (3), the left member is negative; now putting $x = 1$, the left member is positive. Hence the root lies between 1 and 2, and the required root of (1) is 1.1+.

We now repeat these steps and obtain the first decimal of the root of (3), which will be the second decimal of the root of (1), and so on. Indicating the three steps in order by (*a*), (*b*), (*c*), we obtain the successive decimals of the root as shown below, the process of finding the first decimal being included for completeness.

(3) $$x^3 - 10x^2 - 500x + 1000 = 0.$$

(*a*) *Locate the root between* 0 *and* 10.

Neglect terms in x^3 and x^2; then $x = 2$. Try this value and the next smaller value (or larger, if the left member of (3) does not change sign) and the root is located between 1 and 2.

(b) *Diminish roots by figure found in* (a).

$$\begin{array}{rrrr|l}
1 & -10 & -500 & +1000 & \underline{-1} \\
 & -1 & +9 & +509 & \\ \hline
1 & -9 & -509 & +\mathbf{491} & \\
 & -1 & +8 & & \\ \hline
1 & -8 & -\mathbf{517} & & \\
 & -1 & & & \\ \hline
\underline{\underline{1}} & -\mathbf{7} & & &
\end{array}$$

Transformed equation: $x^3 - 7x^2 - 517x + 491 = 0$.

(c) *Multiply roots by* 10.

$$x^3 - 70x - 51{,}700x + 491{,}000 = 0.$$

Repeat these operations on the last equation.

(a) $$x = 491{,}000 \div 51{,}700 = 9+.$$

By trial the sign of the left member is + when x is 9 and 8, but changes when x is 10. Hence the root is between 9 and 10.

(b)

$$\begin{array}{rrrr|l}
1 & -70 & -51{,}700 & +491{,}000 & \underline{-9} \\
 & -9 & +549 & +470{,}241 & \\ \hline
1 & -61 & -52{,}249 & +\mathbf{20{,}759} & \\
 & -9 & +468 & & \\ \hline
1 & -52 & -\mathbf{52{,}717} & & \\
 & -9 & & & \\ \hline
\underline{\underline{1}} & -\mathbf{43} & & &
\end{array}$$

$$x^3 - 43x^2 - 52{,}717x + 20{,}759 = 0.$$

(c) $$x^3 - 430x - 5{,}271{,}700x + 20{,}759{,}000 = 0.$$

The required root of (1) is now $x = 1.19+$. Another repetition of the process gives the third decimal.

(a) $$x = 20{,}759{,}000 \div 5{,}271{,}700 = 4-.$$

The left member has opposite signs for $x = 3$ and $x = 4$, hence the root is between 3 and 4.

(b)

$$\begin{array}{rrrr|l}
1 & -430 & -5{,}271{,}700 & +20{,}759{,}000 & \underline{-3} \\
 & -3 & +1{,}281 & +15{,}818{,}943 & \\ \hline
1 & -427 & -5{,}272{,}981 & +\mathbf{4{,}940{,}057} & \\
 & -3 & +1{,}272 & & \\ \hline
1 & -424 & -\mathbf{5{,}274{,}253} & & \\
 & -3 & & & \\ \hline
\underline{\underline{1}} & -\mathbf{421} & & &
\end{array}$$

$$x^3 - 421x^2 - 5{,}274{,}253x + 4{,}940{,}057 = 0.$$

We thus have the required root of (1) as $x = \mathbf{1.193}+$.

We may omit step (c) in our last operation and get the next figure of the required root by neglecting x^3 and x^2 in the last equation. This gives $x = .9+$, and our root is, finally,

$$x = \mathbf{1.1939+}.$$

A convenient arrangement of the whole operation of finding this root is as follows:

$$\begin{array}{l}
\mathbf{1 - 4 + 0 + 4} \ \lfloor \mathbf{-1} \\
\quad -1 + 3 + 3 \\
\hline
1 - 3 - 3 \mid \mathbf{+1} \\
\quad -1 + 2 \\
\hline
1 - 2 \mid \mathbf{-5} \\
\quad -1 \\
\hline
\underline{1} \mid \mathbf{-1}
\end{array}$$

$$\begin{array}{l}
\mathbf{1 - 10 - 500 + 1000} \ \lfloor \mathbf{-1} \\
\quad -1 + 9 + 509 \\
\hline
1 - 9 - 509 \mid \mathbf{+491} \\
\quad -1 + 8 \\
\hline
1 - 8 \mid \mathbf{-517} \\
\quad -1 \\
\hline
\underline{1} \mid \mathbf{-7}
\end{array}$$

$$\begin{array}{l}
\mathbf{1 - 70 - 51{,}700 + 491{,}000} \ \lfloor \mathbf{-9} \\
\quad -9 + 549 + 470{,}241 \\
\hline
1 - 61 - 52{,}249 \mid \mathbf{+20{,}759} \\
\quad -9 + 468 \\
\hline
1 - 52 \mid \mathbf{-52{,}717} \\
\quad -9 \\
\hline
\underline{1} \mid \mathbf{-43}
\end{array}$$

$$\begin{array}{l}
\mathbf{1 - 430 - 5{,}271{,}700 + 20{,}759{,}000} \ \lfloor \mathbf{-3} \\
\quad -3 + 1{,}281 + 15{,}818{,}943 \\
\hline
1 - 427 - 5{,}272{,}981 \mid \mathbf{+4{,}940{,}057} \\
\quad -3 + 1{,}272 \\
\hline
1 - 424 \mid \mathbf{-5{,}274{,}253} \\
\quad -3 \\
\hline
\underline{1} \mid \mathbf{-421}
\end{array}$$

Root, 1.1939+.

288. In approximating to the roots of an equation, the following remarks should be borne in mind. Let the student supply proofs when needed.

(1) Every equation of odd degree has at least one *real* root. (For $f(x)$ has opposite signs when $x = +\infty$ and $x = -\infty$.)

(2) When an even number of roots lie between $x = a$ and $x = b$, $f(a)$ and $f(b)$ will have like signs.

(3) Whenever more than one root lies between two assumed values of x, especial care must be used to separate them by trial.

(4) The next decimal of a root is obtained approximately by dividing the absolute term of the last transformed equation by the coefficient of x with its sign changed.

(5) Should this decimal be too large, the constant term of the next transformed equation will change sign. (Observe that in the example the constant terms of the original equation and of all the transformed equations are of the same sign.)

(6) Should this decimal be too small, the next transformed equation will not have a root between 0 and 10, except when there happen to be two or more roots of the original equation with the same integral part.

(7) To obtain a negative root, change the signs of all the roots and proceed as for a positive root.

289. Exercises. Calculate to four decimal places the real roots of the equations:

1. $x^3 - 24x - 48 = 0.$

2. $x^3 - 7x^2 + 4x + 24 = 0.$

3. $x^3 - 2x + 1 = 0.$

4. $x^3 - x^2 + x - 1 = 0.$

5. $x^3 + x^2 + x + 1 = 0.$

6. $x^4 - 6x^2 + 5 = 0.$

7. $x^3 - 7x - 5 = 0.$

8. $x^3 - 31x - 19 = 0.$

9. $x^3 - 48x - 112 = 0.$

10. $2x^3 - 18x^2 + 46x - 30 = 0.$

11. $7x^3 - 9x + 5 = 0.$

12. $4x^3 - 3x - 1 = 0.$

13. $x^4 + x^3 - 2x^2 - 3x - 3 = 0.$

14. $x^4 - 2x^3 - 8x^2 + 24x - 48 = 0.$

15. $x^4 - 4x^3 - 8x + 32 = 0.$

16. $x^4 + 2x^3 + x + 2 = 0.$

17. $3x^4 - 2x^3 - 16x^2 - 56x + 96 = 0.$

18. $x^3 - 7x - 7 = 0.$

19. $8x^4 + 16x^3 + 18x^2 + x + 7 = 0.$

20. $7x^3 + 8x^2 - 14x - 16 = 0.$

21. $2x^4 - 5x^3 - 32x + 80 = 0.$

22. $2x^5 - 4x^3 + 3x^2 - 6 = 0.$

290. Cardan's Solution of the Cubic Equation. — As in the case of the quadratic equation, so the equations of third and fourth degree may be solved by means of radicals. This cannot be done for equations of degree higher than the fourth. We give here a solution of the cubic equation

(1) $$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0.$$

We first obtain a new equation containing no term of second degree. To do this, put

$$x = y + h.$$

Expanding and collecting in powers of y,

$$a_0y^3 + 3(a_0h + a_1)y^2 + 3(a_0h^2 + 2a_1h + a_2)y + a_0h^3 + 3a_1h^2 + 3a_2h + a_3 = 0.$$

The term in y^2 drops out if

$$a_0h + a_1 = 0, \quad \text{or} \quad h = -\frac{a_1}{a_0}.$$

With this value of h the equation becomes

$$a_0y^3 + \frac{3(a_0a_2 - a_1^2)}{a_0}y + \frac{a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3}{a_0^2} = 0.$$

Putting $$y = \frac{z}{a_0},$$

we have

$$z^3 + 3(a_0a_2 - a_1^2)z + (a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3) = 0.$$

Let

$$H = a_0a_2 - a_1^2; \quad G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3.$$

Then the equation becomes

(2) $$z^3 + 3Hz + G = 0.$$

To solve this equation let

$$z = \sqrt[3]{r} + \sqrt[3]{s}.$$

Then

$$z^3 = r + s + 3\sqrt[3]{rs}\left(\sqrt[3]{r} + \sqrt[3]{s}\right),$$

or, $$z^3 - 3\sqrt[3]{rs} \cdot z - (r + s) = 0.$$

If this is to be identical with (2), we must have

$$\sqrt[3]{rs} = -H, \quad \text{and} \quad r + s = -G;$$

or, $$rs = -H^3, \quad \text{and} \quad r + s = -G.$$

Solving for r and s,

$$r = \frac{-G + \sqrt{G^2 + 4H^3}}{2}; \quad s = \frac{-G - \sqrt{G^2 + 4H^3}}{2}.$$

Then

$$z = \sqrt[3]{r} + \sqrt[3]{s} = \sqrt[3]{r} - \frac{H}{\sqrt[3]{r}}. \quad (rs = -H^3.)$$

Let the three cube roots of r be α_1, α_2, and α_3. Then the three values of z are

$$z_1 = \alpha_1 - \frac{H}{\alpha_1}; \quad z_2 = \alpha_2 - \frac{H}{\alpha_2}; \quad z_3 = \alpha_3 - \frac{H}{\alpha_3}.$$

The corresponding values of x are then found from

$$x = y + h = y - \frac{a_1}{a_0} = \frac{z}{a_0} - \frac{a_1}{a_0} = \frac{z - a_1}{a_0}.$$

Nature of the Roots. — The following criteria serve to determine the nature of the roots:

(a) $G^2 + 4H^3 < 0$, three real distinct roots;

(b) $G^2 + 4H^3 = 0$, three real roots, two being equal;

(c) $G^2 + 4H^3 > 0$, one real root, two imaginary roots.

By direct calculation, for which we shall not take space, we find

$$(z_1 - z_2)(z_2 - z_3)(z_3 - z_1) = \sqrt{-27(G^2 + 4H^3)},$$

or,

$$(z_1 - z_2)^2 (z_2 - z_3)^2 (z_3 - z_1)^2 = -27(G^2 + 4H^3).$$

When the roots are all real, their differences are real, hence the left member of the last equation is positive; therefore $G^2 + 4H^3$ must be negative. When two roots are equal, their difference is zero; hence $G^2 + 4H^3 = 0$. When two roots are imaginary, they must be conjugate imaginaries; suppose them to be

$$z_1 = a + ib \quad \text{and} \quad z_2 = a - ib.$$

Let the third root be $z_3 = c$, where c is real [(1), (**288**)]. Then we show directly that $(z_1 - z_2)^2$ is negative, and that $(z_2 - z_3)^2(z_3 - z_1)^2$ is positive, hence the left member of the above equation is negative; therefore $G^2 + 4H^3$ must be positive.

The quantity $G^2 + 4H^3$ is called the *discriminant* of the cubic

$$z^3 + 3Hz + G = 0.$$

When all the roots are real, i.e., $G^2 + 4H^3 < 0$, r and s are conjugate complex quantities; let them be

$$r = A + iB; \quad s = A - iB.$$

In this case $\sqrt[3]{r}$ and $\sqrt[3]{s}$ cannot be evaluated algebraically. The roots may then be obtained in trigonometric form. Let

$$A = u \cos v; \quad B = u \sin v.$$

Then

$$r = u(\cos v + i \sin v); \quad s = u(\cos v - i \sin v).$$

Hence

$$\sqrt[3]{r} = \sqrt[3]{u}\left(\cos\frac{v + 2k\pi}{3} + i \sin\frac{v + 2k\pi}{3}\right),$$

$$\sqrt[3]{s} = \sqrt[3]{u}\left(\cos\frac{v + 2k\pi}{3} - i \sin\frac{v + 2k\pi}{3}\right); \quad k = 0, 1, 2.$$

Here $\sqrt[3]{u}$ denotes the real cube root of u.

We now find

$$z = \sqrt[3]{r} + \sqrt[3]{s} = 2\sqrt[3]{u}\cos\frac{v + 2k\pi}{3}; \quad k = 0, 1, 2.$$

291. Ferrari's Solution of the Quartic Equation. — Write the given quartic equation in the form

(1) $$x^4 + 2ax^3 + bx^2 + 2cx + d = 0.$$

Add to both members $(px + q)^2$:

(2) $$x^4 + 2ax^3 + (b + p^2)x^2 + 2(c + pq)x + (d + q^2) = (px + q)^2.$$

The left member will become a perfect trinomial square of the form

$$(x^2 + ax + k)^2$$

by putting

(3) $$p^2 = a^2 - b + 2k; \quad q^2 = -d + k^2; \quad pq = -c + ak.$$

Then equation (2) becomes

$$(x^2 + ax + k)^2 = (px + q)^2,$$

or,

(4) $$x^2 + ax + k = \pm(px + q).$$

Taking each sign in turn we have two quadratic equations in x, which give the four roots of (1).

To obtain the values of p, q, and k in (4) we must solve equations (3) for these quantities in terms of the coefficients. On equating the values of p^2q^2 from the product of the first two of equations (3) and the square of the third equation we find a cubic to determine k:

$$(5) \qquad 2k^3 - bk^2 + 2(ac - d)k + (bd - a^2d - c^2) = 0.$$

This is called the *reducing cubic*, and is to be solved for a real value of k. Then p and q are obtained from (3).

Example. $\quad x^4 + 4x^3 - 3x^2 - 16x + 5 = 0.$

Here $\quad a = 2,\ b = -3,\ c = -8,\ d = 5.$

Then (5) is $\quad 2k^3 + 3k^2 - 42k - 99 = 0.$

A real root is $\quad k = -3.$

Then from (3), $\quad p = 1,\ q = 2;\ \text{ or, }\ p = -1,\ q = -2.$

With either set of values of p and q (4) becomes

$$(x^2 + 2x - 3) = \pm(x + 2).$$

Hence

$$x^2 + x - 5 = 0, \quad \text{or,} \quad x^2 + 3x - 1 = 0.$$

Therefore

$$x = \frac{-1 \pm \sqrt{21}}{2}, \quad \text{or,} \quad \frac{-3 \pm \sqrt{13}}{2}.$$

Exercises. Solve the following equations:

1. $x^3 - 3x^2 + 4 = 0.$

2. $x^3 - 3x - 2 = 0.$

3. $4x^3 - 3x - 1 = 0.$

4. $x^3 - 24x - 48 = 0.$

5. $x^3 - 7x^2 + 4x + 24 = 0.$

6. $x^3 - 3x^2 - 6x + 1 = 0.$

7. $x^3 - 7x - 6 = 0.$

8. $x^3 - x^2 + x - 1 = 0.$

9. $x^4 + 2x^3 + 2x^2 - 2x - 3 = 0.$

10. $x^4 + 6x^3 + 3x^2 - 2x - 3 = 0.$

11. $x^4 - 4x^3 - 9x^2 + 2x + 3 = 0.$

12. $x^4 + 4x^3 - 16x + 11 = 0.$

13. $x^4 + 4x^3 - 16x - 16 = 0.$

14. $x^4 - 3x^3 - 7x^2 + 15x + 18 = 0.$

15. $x^4 - 4x^3 - 8x + 32 = 0.$

16. $x^4 + x^3 - 2x^2 - 3x - 3 = 0.$

CHAPTER XX

Spherical Trigonometry

292. Spherical Geometry. — We devote this article to a review of some facts concerning the geometry of the sphere.

(a) A plane section of a sphere is a circle. When the plane passes through the center, the section is a *great circle;* otherwise a *small circle.*

(b) Any two great circles intersect in two diametrically opposite points and bisect each other.

(c) The two points on the sphere each equally distant from all the points of a circle on the sphere are called the *poles* of the circle. A great circle is 90° distant from each of its poles.

(d) A *spherical triangle* is a figure bounded by three circular arcs on a sphere. In this chapter we consider only triangles whose sides are arcs of great circles. Any such triangle may therefore be considered as cut from the spherical surface by the faces of a triedral angle whose vertex is at the center. The face angles of this triedral angle measure the sides of the triangle, and its diedral angles the angles of the triangle.

(e) If a triangle be constructed by striking arcs with the vertices of a given triangle as poles, the new triangle is called the *polar triangle* of the given one.

Let the sides of the given triangle be a, b, c; its angles A, B, C; let the sides of the polar triangle be a', b', c' and its angles A', B', C'; we assume that A is the pole of a', B of b', and C of c'; then

$$a' = 180 - A\ ; \qquad A' = 180 - a\ ;$$

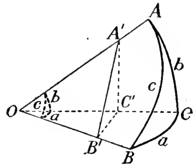
and similarly for the other sides and angles. That is, *any part of the polar triangle is the supplement of the part opposite in the given triangle.*

(f) The difference between the sum of the angles of a spherical triangle and 180° is called its *spherical excess.*

The area of a spherical triangle is to the area of the sphere as its spherical excess, in degrees, is to 720°. That is, if E be the spherical excess in degrees and K the area, and R the radius of the sphere, then

$$K = \frac{E}{720} \times 4\pi R^2.$$

293. Spherical Right Triangles. — Let O be the center of a sphere and ABC a triangle on its surface having $C = 90°$. The triangle shown in the figure has each part, except C, less than 90°. The results below are true in any case, as may be shown by drawing other figures, or by assuming the right triangle as a special case of the oblique triangle.



Cut the triedral angle $O-ABC$ by a plane $\perp OB$, forming the plane right $\triangle$ $A'B'C'$, with $C' = 90°$. Then also $\triangle$s $OB'C'$ and $OB'A'$ are right-angled at B'. Further, $\angle A'B'C'$ measures $\angle B$ (**292**, (*d*)). Then

(a) $$\sin B = \sin A'B'C' = \frac{A'C'}{A'B'} = \frac{\frac{A'C'}{OA'}}{\frac{A'B'}{OA'}} = \frac{\sin b}{\sin c}.$$

(b) $$\cos B = \cos A'B'C' = \frac{B'C'}{A'B'} = \frac{\frac{B'C'}{OB'}}{\frac{A'B'}{OB'}} = \frac{\tan a}{\tan c}.$$

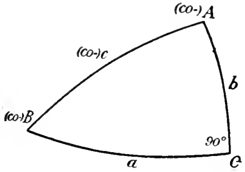
(c) $$\tan B = \tan A'B'C' = \frac{A'C'}{B'C'} = \frac{\frac{A'C'}{OC'}}{\frac{B'C'}{OC'}} = \frac{\tan b}{\sin a}.$$

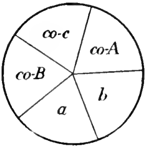
Dividing (a) by (b) and comparing with (c) we have

(d) $$\cos c = \cos a \cos b.$$

By combining these equations we may obtain others by which any part of the triangle may be expressed directly in terms of any two given parts, the right angle excluded. These formulas are all contained in two simple rules.

294. Napier's Rules of Circular Parts. — Let co-x denote the complement of any part x of the triangle. Take the complements of c, A, B, and arrange the five parts, a, b, co-A, co-c, co-B, called *circular parts* in the order in which they occur in the triangle as in the adjacent figures. Then if any one of the five be taken as the middle part, of the other four parts two will be *adjacent* and





the other two *opposite* to this part. Thus, if co-c be taken as the middle part, co-B and co-A are adjacent, a and b opposite.

Rules:

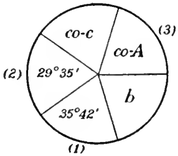
Sine of Middle Part = { *Product of tangents of adjacent parts,* *or* *Product of cosines of opposite parts.* }

Exercise. Taking each part in turn as the middle part write out a complete list of formulas relating to the spherical right triangle. Derive these formulas from those given above.

295. Solution of Right Triangles.

Example. Given $a = 35° 42'$; $B = 60° 25'$. Find b, c, A.

The diagram of circular parts is shown in the figure. Taking (1), (2), (3) in turn as middle part we have



(1) $\sin 35° 42' = \tan 29° 35' \tan b$;

(2) $\sin 29° 35' = \tan 35° 42' \tan (\text{co-}c)$;

(3) $\sin (\text{co-}A) = \cos 29° 35' \cos 35° 42'$.

Hence,

$$\tan b = \frac{\sin 35° 42'}{\tan 29° 35'};\quad \cot c = \frac{\sin 29° 35'}{\tan 35° 42'};$$
$$\cos A = \cos 29° 35' \cos 35° 42'.$$

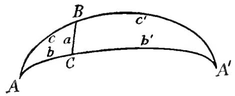
Check. The computed parts must satisfy the relation

$$\sin (\text{co-}A) = \tan b \tan (\text{co-}c), \text{ or } \cos A = \tan b \cot c.$$

Computations.

log	log	log
$\sin 35° 42' = 9.7660$	$\sin 29° 35' = 9.6934$	$\cos 29° 35' = 9.9394$
$\tan 29° 35' = 9.7541$	$\tan 35° 42' = 9.8564$	$\cos 35° 42' = 9.9096$
$\tan b = 0.0119$	$\cot c = 9.8370$	$\cos A = 9.8490$
$b = 45° 17'$	$c = 55° 30'$	$A = 45° 4'$

Check. $\log \cos A = \log \tan b + \log \cot c.$
$$9.8490 = 0.0119 + 9.8370.$$



Ambiguous Case. When the given parts are an angle (not the right angle) and its opposite side, two solutions are possible, because the other parts are then calculated from their sines. The two triangles together form a lune, as AA' in the figure, where A, a are supposed to be the given parts.

296. Quadrantal Triangles. — A quadrantal triangle is one having a side equal to a quadrant or 90°. Its polar triangle will be a right triangle, which may be solved by Napier's Rules. The parts of the given quadrantal triangle then become known by (e) of (**292**).

Exercises. Solve the following triangles, C being the right angle:

1. $a = 45° 10'$, $B = 70° 20'$.	**4.** $b = 100°$, $a = 40°$.	**7.** $B = 145° 53'$, $c = 110° 20'$.
2. $b = 65° 15'$, $A = 25° 50'$.	**5.** $A = 120° 42'$, $c = 56° 50'$.	**8.** $b = 132° 16'$, $B = 65° 46'$.
3. $c = 33° 18'$, $b = 30° 37'$.	**6.** $A = 40°$, $a = 30°$.	**9.** $c = 170° 4'$, $a = 175° 17'$.

Solve the following quadrantal triangles:

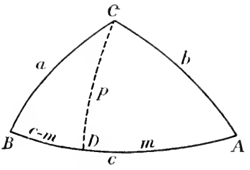
10. $a = 90°$, $b = 50°$, $c = 40°$.	**11.** $A = 65° 15'$, $b = 90°$, $c = 50° 25'$.	**12.** $A = 122° 10'$, $B = 70° 22'$, $c = 90°$.

297. Oblique Triangles. Two Fundamental Formulas. — We consider only triangles in which no part exceeds 180°.

I. Law of Sines. — Let ABC be a spherical triangle. Draw $CD \perp AB$, forming two right triangles (figure).

In $\triangle ACD$, $\sin p = \sin b \sin A$.

In $\triangle BCD$, $\sin p = \sin a \sin B$.



Therefore,

$$\sin b \sin A = \sin a \sin B, \text{ or}$$

(1) $$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}.$$

That is, *the sines of the sides are proportional to the sines of the opposite angles.*

Exercise. Discuss the case in which D falls on AB produced.

II. Law of Cosines. — In the figure above let $AD = m$, so that $BD = c - m$. Then in right $\triangle BCD$

$$\cos a = \cos (c - m) \cos p, \quad . \; . \; . \quad \text{(d), } \mathbf{(293)}$$
$$= \cos c \cos m \cos p + \sin c \sin m \cos p.$$

But in $\triangle ACD$

$$\cos m \cos p = \cos b$$

and $$\sin m \cos p = \sin C \sin b \times \frac{\cos A}{\sin C} = \sin b \cos A.$$

Hence

(2) $$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

That is, *the cosine of any side equals the product of the cosines of the other two sides plus the product of their sines by the cosine of their included angle.*

Exercise. Discuss the case where D falls on AB produced.

From the fundamental formulas (**1**) and (**2**) we shall derive a series of other formulas adapted to the solution of triangles.

298. Principle of Duality. — By means of (e) of (**292**) any formula relating to the spherical triangle can be made to yield a second formula. Thus, let $\triangle A'B'C'$ be polar to $\triangle ABC$. Then from (1) and (2)

$$\frac{\sin a'}{\sin b'} = \frac{\sin A'}{\sin B'}; \qquad \cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

But $$a' = 180 - A, \quad A' = 180 - a,$$
$$b' = 180 - B', \quad B' = 180 - b,$$
$$c' = 180 - C', \quad C' = 180 - c.$$

Substituting and reducing, we have

$$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b},$$

(3) $$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

The first of these is simply the law of sines; the second is a new formula.

299. Formulas for the Half Angle. — Solving (2) for $\cos A$, we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Then

$$\sin \frac{1}{2} A = \sqrt{\frac{1 - \cos A}{2}} \quad \left(\text{Why not} \pm \sqrt{\frac{1 - \cos A}{2}}?\right)$$

$$= \sqrt{\frac{1 - \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}}{2}}$$

$$= \sqrt{\frac{\sin b \sin c - \cos a + \cos b \cos c}{2 \sin b \sin c}}$$

$$= \sqrt{\frac{\cos (b - c) - \cos a}{2 \sin b \sin c}}$$

$$= \sqrt{\frac{2 \sin \dfrac{a + b - c}{2} \sin \dfrac{a - b + c}{2}}{2 \sin b \sin c}}.$$

Now let

(4) $$2s = a + b + c;$$

then

$$\frac{a + b - c}{2} = s - c \quad \text{and} \quad \frac{a - b + c}{2} = s - b;$$

therefore,

(5) $$\sin \frac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}.$$

Similarly,

(6) $$\cos \frac{1}{2} A = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}.$$

By dividing,

(7) $$\tan \frac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}.$$

Given the three sides, one of these formulas, preferably the last, will determine the angles. When all three angles are desired, let

(8) $$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$$

then

(9) $$\tan\frac{1}{2}A = \frac{\tan r}{\sin(s-a)},$$

(10) $$\tan\frac{1}{2}B = \frac{\tan r}{\sin(s-b)},$$

(11) $$\tan\frac{1}{2}C = \frac{\tan r}{\sin(s-c)}.$$

300. Formulas for the Half Sides. — Proceeding as above with (3) of (**298**), or by applying the principle of duality to formulas (5) to (11) we have, on putting

(12) $$2S = A + B + C$$

and

(13) $$\tan R = \sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}},$$

(14) $$\sin\frac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}},$$

(15) $$\cos\frac{1}{2}a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}},$$

(16) $$\tan\frac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}},$$

(17) $$\tan\tfrac{1}{2}a = \tan R\cos(S-A),$$

(18) $$\tan\tfrac{1}{2}b = \tan R\cos(S-B),$$

(19) $$\tan\tfrac{1}{2}c = \tan R\cos(S-C).$$

301. Napier's Analogies. — Dividing $\tan\frac{1}{2}A$ by $\tan\frac{1}{2}B$ and reducing, we have

$$\frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}B} = \frac{\sin(s-b)}{\sin(s-a)}.$$

By composition and division,

$$\frac{\tan\frac{1}{2}A + \tan\frac{1}{2}B}{\tan\frac{1}{2}A - \tan\frac{1}{2}B} = \frac{\sin(s-b) + \sin(s-a)}{\sin(s-b) - \sin(s-a)}.$$

Reducing tangents to sines and cosines and simplifying the resulting complex fraction, applying the formulas for $\sin(x \pm y)$ on the left and for $\sin u \pm \sin v$ on the right, we have

$$\text{(20)} \qquad \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)} = \frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)},$$

or,

$$\text{(20}') \qquad \tan\frac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{1}{2}c.$$

Multiplying $\tan\frac{1}{2}A$ by $\tan\frac{1}{2}B$ and reducing,

$$\frac{\tan\frac{1}{2}A\tan\frac{1}{2}B}{1} = \frac{\sin(s-c)}{\sin c}.$$

By composition and division, and reduction as above,

$$\text{(21)} \qquad \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} = \frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)},$$

or,

$$\text{(21}') \qquad \tan\frac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{1}{2}c.$$

These formulas determine the other two sides when two angles and their included side are given.

Proceeding as above with $\tan\frac{1}{2}a$ and $\tan\frac{1}{2}b$, or by the principle of duality applied to formulas (20) to (21′), we obtain

$$\text{(22)} \qquad \frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)} = \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)},$$

or,

$$\text{(22}') \qquad \tan\frac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{1}{2}C,$$

$$\text{(23)} \qquad \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)} = \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A+B)},$$

or,

$$\text{(23}') \qquad \tan\frac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{1}{2}C.$$

These formulas determine the other two angles when two sides and their included angle are given.

302. Area of a Spherical Triangle. — This may be calculated by (f) of (**312**), namely,

$$K = \frac{E \text{ (degrees)}}{720} \times 4\,\pi\,R^2, \quad \text{or,} \quad K = E \text{ (radians)} \times R^2.$$

To obtain E, we may first calculate the angles. E may also be obtained by one of the following formulas, which we add without proofs.

$$\tan\frac{1}{2}E = \frac{\tan\frac{1}{2}a\tan\frac{1}{2}b\sin C}{1+\tan\frac{1}{2}a\tan\frac{1}{2}b\cos C};$$

$$\tan\frac{1}{4}E = \sqrt{\tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}}.$$

303. Solution of Spherical Oblique Triangles. — Six cases arise, according to the nature of the three given parts.

I. Given two sides and an opposite angle.

Denote the given parts by a, b, A. Calculate B by (1), then C by (22) or (23), and c by (20) or (21).

Check: $$\frac{\sin b}{\sin c} = \frac{\sin B}{\sin C},$$

which involves the *computed parts.*

Ambiguous Case. Formula (1) will give two (supplementary) values for B. Two solutions are obtained when both values of B lead to values of C. Otherwise one or both values of B must be rejected.

Rule. Retain values of B which make $A - B$ and $a - b$ of like sign. Otherwise (20) and (22) take the impossible form $+ = -$.

II. Given two angles and an opposite side.

Denote the given parts by A, B, a. Calculate b by (1), then proceed as in **I.**

Ambiguous Case. Formula (1) gives two values of b. *Retain the value or values which make $A - B$ and $a - b$ of like sign.*

III. Given the three sides.

Calculate the angles by (9), (10), (11).

Check: $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

IV. Given the three angles.

Calculate the sides by (17), (18), (19).

Check: As in **III.**

V. Given two sides and their included angle.

Denote the given parts by a, b, C. Calculate $\frac{1}{2}(A+B)$ by (23′), $\frac{1}{2}(A-B)$ by (22′); then A and B by addition and subtraction; obtain c by the law of sines. Check by (20) or (21).

VI. Given two angles and their included side.

Denote the given parts by A, B, c. Calculate $\frac{1}{2}(a+b)$ from (21′), $\frac{1}{2}(a-b)$ from (20′); hence get a and b; obtain C by the law of sines. Check by (22) or (23).

304. Example. Given $a = 100°\,37'$, $b = 62°\,25'$, $A = 120°\,48'$.

Formulas.

$$\sin B = \frac{\sin b}{\sin a}\sin A,$$

$$\cot\tfrac{1}{2}C = \frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}\tan\tfrac{1}{2}(A-B),$$

$$\tan\tfrac{1}{2}c = \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}\tan\tfrac{1}{2}(a-b).$$

Check:
$$\frac{\sin b}{\sin c} = \frac{\sin B}{\sin C}.$$

Computations.

$\log\sin b = 9.9476$	$a = 100°\,37'$	$A = 120°\,48'$
$\log\sin A = 9.9340$	$b = 62°\,25'$	$B = 50°\,46'$
$\text{colog}\sin a = 0.0075$	$a+b = 162°\,62'$	$A+B = 170°\,94'$
$\log\sin B = 9.8891$	$a-b = 38°\,12'$	$A-B = 70°\,2'$
$B = 50°\,46'.5$	$\frac{1}{2}(a+b) = 81°\,31'$	$\frac{1}{2}(A+B) = 85°\,47'$
or $129°\,13'.5$	$\frac{1}{2}(a-b) = 19°\,6'$	$\frac{1}{2}(A-B) = 35°\,1'$

Reject the larger value of B by the rule in **I.**

$\log\tan\frac{1}{2}(A-B) = 9.8455$	$\log\tan\frac{1}{2}(a-b) = 9.5395$
$\log\sin\frac{1}{2}(a+b) = 9.9952$	$\log\sin\frac{1}{2}(A+B) = 9.9989$
$\text{colog}\sin\frac{1}{2}(a-b) = 0.4852$	$\text{colog}\sin\frac{1}{2}(A-B) = 0.2412$
$\log\cot\frac{1}{2}C = 0.3259$	$\log\tan\frac{1}{2}c = 9.7796$
$\frac{1}{2}C = 64°\,43'.5$	$\frac{1}{2}c = 31°\,3'$
$C = 129°\,27'$	$c = 62°\,6'$
Check: $\log\sin b = 9.9476$	$\log\sin B = 9.8891$
$\sin c = 9.9463$	$\sin C = 9.8877$
0.0013	0.0014

Note. In the solutions of triangles, a complete form should be prepared in advance, so that only numerical values need be inserted when the tables are opened.

305. Exercises. Solve the triangles whose given parts are:

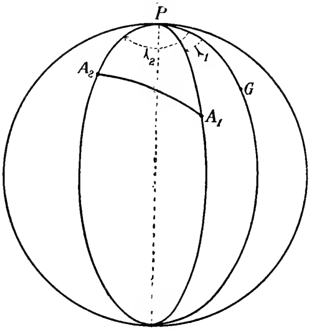
1.	**2.**	**3.**	**4.**
$a = 53^\circ\ 18'.3$,	$a = 42^\circ\ 15'.3$,	$a = 84^\circ\ 14'\ 30''$,	$A = 116^\circ\ 8'.5$,
$b = 36^\circ\ 5'.6$,	$b = 33^\circ\ 18'.8$,	$b = 44^\circ\ 13'\ 46''$,	$B = 35^\circ\ 46'.6$,
$c = 50^\circ\ 24'.9$.	$c = 60^\circ\ 32'.1$.	$c = 51^\circ\ 6'\ 20''$.	$C = 46^\circ\ 33'.7$.
5.	**6.**	**7.**	**8.**
$A = 97^\circ\ 53'$,	$A = 53^\circ\ 42'\ 34''$,	$a = 89^\circ\ 0'$,	$a = 70^\circ\ 20'$,
$B = 67^\circ\ 59'.7$,	$B = 62^\circ\ 24'\ 26''$,	$b = 47^\circ\ 30'$,	$b = 38^\circ\ 28'$,
$C = 84^\circ\ 46'.7$.	$C = 155^\circ\ 43'\ 12''$.	$C = 36^\circ\ 0'$.	$C = 52^\circ\ 30'$.
9.	**10.**	**11.**	**12.**
$b = 19^\circ\ 24'$,	$a = 88^\circ\ 24'\ 3''$,	$a = 102^\circ\ 22'$,	$b = 76^\circ\ 40'\ 48''$,
$c = 41^\circ\ 36'$,	$c = 120^\circ\ 10'\ 55''$,	$B = 84^\circ\ 30'$,	$A = 84^\circ\ 30'\ 20''$,
$A = 84^\circ\ 10'$.	$B = 49^\circ\ 27'\ 50''$.	$C = 125^\circ\ 28'$.	$C = 130^\circ\ 51'\ 33''$.
13.	**14.**	**15.**	**16.**
$c = 104^\circ\ 13'.4$,	$c = 108^\circ\ 39'\ 10''$,	$b = 54^\circ\ 18'\ 16''$,	$a = 88^\circ\ 27'\ 50''$,
$A = 63^\circ\ 48'.6$,	$A = 64^\circ\ 48'\ 52''$,	$A = 127^\circ\ 22'\ 7''$,	$b = 107^\circ\ 19'\ 52''$,
$B = 51^\circ\ 46'.2$.	$B = 40^\circ\ 23'\ 17''$.	$C = 72^\circ\ 26'\ 40''$.	$C = 116^\circ\ 15'\ 0''$.
17.	**18.**	**19.**	**20.**
$b = 83^\circ\ 5'\ 36''$,	$b = 68^\circ\ 45'$,	$a = 56^\circ\ 37'$,	$a = 48^\circ$,
$c = 64^\circ\ 3'\ 20''$,	$B = 58^\circ\ 5'$,	$A = 123^\circ\ 54'$,	$b = 67^\circ$,
$A = 57^\circ\ 50'\ 0''$.	$C = 50^\circ\ 51'$.	$B = 57^\circ\ 47'$.	$A = 42^\circ$.
21.	**22.**	**23.**	**24.**
$b = 81^\circ$,	$a = 69^\circ\ 34'.9$,	$a = 69^\circ\ 11'.8$,	$a = 151^\circ\ 01'\ 5''$,
$A = 72^\circ$,	$c = 70^\circ\ 20'.3$,	$b = 56^\circ\ 38'.5$,	$b = 134^\circ\ 10'\ 52''$,
$B = 119^\circ$.	$C = 50^\circ\ 30'.1$.	$A = 68^\circ\ 40'$.	$A = 144^\circ\ 20'\ 45''$.
25.	**26.**	**27.**	**28.**
$a = 40^\circ\ 8'\ 28''$,	$a = 88^\circ\ 12'.3$,	$c = 100^\circ\ 49'\ 30''$,	$A = 45^\circ$,
$b = 118^\circ\ 20'\ 8''$,	$A = 63^\circ\ 15'.2$,	$B = 95^\circ\ 38'\ 11''$,	$a = 10^\circ$,
$A = 29^\circ\ 45'\ 32''$.	$B = 132^\circ\ 18'$.	$C = 97^\circ\ 26'\ 28''$.	$b = 60^\circ$.

306. Applications to the Terrestrial Sphere. — We shall consider the earth as a sphere with a radius of 3960 miles. Longitudes are to be reckoned from Greenwich westward through 360° or 24^h. We shall denote longitude by λ, latitude by ϕ.

Problem 1. Given the latitudes and longitudes of two stations, to find the distance between them.

Let P be the earth's north pole, G Greenwich, A_1 and A_2 the two stations (figure). Let the positions of the two stations be λ_1, ϕ_1 and λ_2, ϕ_2 respectively.

Then in $\triangle A_1PA_2$, $PA_1 = 90° - \phi_1$, $PA_2 = 90° - \phi_2$, and $\angle A_1PA_2 = \lambda_2 - \lambda_1$. Hence in $\triangle A_1PA_2$ two sides and their included angle are known, and A_1A_2 (in degrees) may be calculated as in **V** of (**303**).



Problem 2. A ship is to sail from A_1 to A_2 by the shortest path (great circle). On what course (at what angle with the meridian) will she depart from A_1; on what course will she arrive at A_2?

Assuming the positions of A_1 and A_2 given, we have two sides and the included angle of the triangle A_1PA_2. We must calculate angles A_1 and A_2. This comes under **V** of (**303**).

Exercises.

1. Calculate the sides (in miles), the angles, and the area (in square miles) of the triangle whose vertices are:

	h m s	
New York	λ = 4 55 54,	ϕ = 40° 45′ N.
San Francisco	8 9 43,	37° 47′ N.
Mexico City	6 36 27,	19° 26′ N.

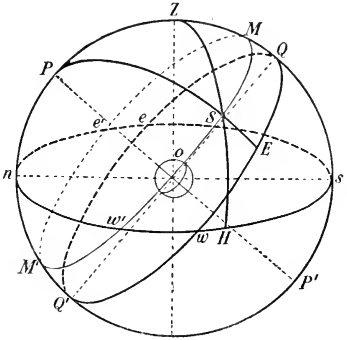
2. A vessel sails on a great circle from San Francisco, $\lambda = 8^h\ 9^m\ 43^s$, $\phi = 37°\ 47'$ N. to Sydney, $\lambda = 13^h\ 55^m\ 10^s$, $\phi = 33°\ 52'$ S. Find the courses of departure and arrival and the distance sailed.

3. If the vessel in exercise 2 makes 12 knots an hour, what is her position (λ and ϕ) and on what course is she sailing 5 days after leaving San Francisco? (1 knot = 1 nautical mile = 1′ on a great circle.)

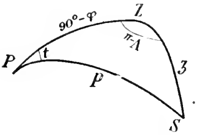
307. Applications to the Celestial Sphere. — For the purpose of this article we assume the *celestial sphere* to be an indefinitely large sphere concentric with that of the earth. On it as a background we see all celestial objects.

The projections on the celestial sphere of the earth's poles, equator, meridians and parallels of latitude are named respectively the *celestial poles* (P, P' in the figure), the *celestial equator* or simply *equator* ($QwQ'e$), *hour circles* (as PSE), and *parallels of declination* (as MSM').

An observer at O on the earth's surface will have his *zenith* at Z, where the plumb line at O, if produced, would meet the celestial sphere; his *horizon* is the great circle *swne*, whose pole is Z; his *meridian* is the great circle $nPZQs$, meeting the horizon in the north and south points.



Let S be a point on the celestial sphere, as the sun's center, or a star. Because of the rotation of the earth, S will appear to describe the parallel $e'MSw'M'e'$, rising at e' and setting at w'. When S has the position shown in the figure, HS is its *altitude*, denoted by h (height above horizon); $\angle\ sZH$ (measured by arc sH) is its *azimuth*, denoted by A; ZS, or $90° - h$, is the *zenith distance* of S and denoted by z. Thus h and A, or z and A, completely define the position of S with reference to horizon and zenith.



With reference to the equator and pole, ES is called the *declination* of S, denoted by δ, and $\angle\ QPE$ (angle which hour circle PS of S makes with meridian PQ) is called its *hour angle*, denoted by t; PS or $90° - \delta$ is the *polar distance* of S, and denoted by p. Thus the position of S is defined by δ and t, or by p and t.

$\triangle\ PZS$ is called the *astronomical triangle;* its parts, except the angle at S which we shall not need, are:

$$PZ = 90° - nP = 90° - \phi; \qquad (\phi = \text{latitude of } O.)$$
$$PS = p = 90° - \delta; \qquad ZS = z = 90° - h;$$
$$\angle\ ZPS = t; \qquad \angle\ PZS = 180° - A.$$

Problem 1. Given the latitude of O, and the declination and altitude of S, calculate the hour angle and azimuth of S.

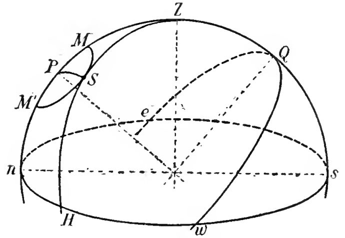
Here the three sides of $\triangle PZS$ are known, and it is only necessary to calculate the angles at P and Z (**III, 303**).

Problem 2. In a given latitude, and for a given declination of the sun, find the sun's hour angle at sunset and the length of day (sunrise to sunset).

Here S is on the horizon and PZS a quadrantal triangle. We obtain t by solving the polar right triangle for $180 - t$. The length of day will be $2\,t$.

Problem 3. Given the sun's declination and its hour angle when it bears due west ($A = 90°$), find the latitude.

Here PZS is a right triangle, with the right angle at Z; p and t are known, and PZ may be calculated by use of Napier's Rules.



Problem 4. Find the hour angle and azimuth of Polaris when at greatest elongation, given the declination of the star and the latitude of the station of observation.

Let MSM' be the star's diurnal path about the pole (figure). When the star is at *greatest elongation*, the great circle ZS is tangent to the small circle MSM', of which PS is a radius. Hence $\triangle PZS$ is right-angled at S; PZ and PS are known, and the angles at P and Z may be found by aid of Napier's Rules.

Exercises.

1. In latitude 40° 49′ the sun's altitude is observed to be 20° 20′; its declination is 15° 12′; find its azimuth and hour angle.

2. With latitude and declination as in exercise 1, find the sun's hour angle when it is due west; when it sets; find its azimuth at sunset; find the length of day.

3. With latitude and declination as in exercise 1, find the sun's altitude and azimuth when its hour angle is 45°.

4. The sun, in declination 12° 22′, is observed to have an altitude of 30° when due west. What is the latitude of the station?

5. The declination of Polaris being 88° 49′, find his azimuth and hour angle at greatest elongation at a station in latitude 40° 49′.

6. As in exercise 5 for the star 51 Cephei, $\delta = 87° 11'$, and for δ Ursæ Minoris, $\delta = 86° 37'$.

7. The stylus of a horizontal sundial consists of a rod pointing to the north celestial pole. Hence its shadow falls due north when the sun is on the meridian, that is, at apparent noon. What angle does its shadow make with the meridian one hour after apparent noon, at a place in latitude 40°?

(*Suggestion.* In the first figure of this article let $nP = 40°$ and $\angle ZPS = 1^h$ or 15°. The stylus lies in the line $P'P$, and its shadow, cast by the sun S, must lie in the plane $SP'P$, and hence will fall on the plane of the dial, *swne*, along the line of intersection of these two planes. This line will be determined by the center of the sphere and the point where arc SP produced will meet arc *ne*. Call this point S'. Then arc nS' measures the required angle, and may be found by solving right $\triangle nPS'$, in which $nP = 40°$ and $\angle nPS' = 15°$).

8. What angle does the shadow of a horizontal sundial make with its noon position t hours after noon in latitude ϕ? (*Ans.* $\tan x = \tan t \sin \phi$, x being the required angle.)

9. Calculate the angles which the hour lines of a horizontal sundial make with the noon-line in an assumed latitude.

ANSWERS

(Answers are given only for the odd-numbered exercises.)

Article 10

1. $\frac{1}{4}a - \frac{1}{4}$. **3.** $.05\,a^2 - 3\,ab - 4.625\,ac$. **5.** $63\,x - 2\,y - 4\,z$. **7.** $a^2b^2c - \frac{8}{3}\,a^2c^4 + \frac{4}{15}\,a^3cd^2 - 2\,a^2c$. **9.** $1.2\,a^3bc^2d^4 - 1.8\,ac^3d^7 + .3\,a^2c^5d^4 - 3\,ac^2d^4$. **11.** $x^6 - 5\,x^4 + 3\,x^3 + 6\,x^2 - 7\,x + 2$. **13.** $x^7 - 9\,x^5y^2 + 7\,x^4y^3 + 13\,x^3y^4 - 19\,x^2y^5 + 8\,xy^6 - y^7$. **15.** $\frac{1}{2}\,a^2 + \frac{2}{27}$. **17.** $x^4 - a^2x^2 - b^2x^2 + a^2b^2$. **19.** $9\,a^2 - 9\,a + 6$. **21.** $-3\,m^2p$. **23.** $-\frac{30}{31}\,xy^2$. **25.** $\frac{.07\,(x^4 - 2\,x^2 + 1)}{x^2}$. **27.** $-\frac{10\,x}{y^2} + \frac{25\,y}{24\,x} + \frac{5\,y^2}{12\,x^2} - \frac{5}{2\,x^2y}$. **29.** $3\,a^2x - 4\,ax^2 + x^3$. **31.** $x^2 + 5\,xy + 3\,y^2$. **33.** $\frac{3}{8}\,a^3 - \frac{2}{3}\,a^2b + \frac{1}{2}\,ab^2$. **35.** $x^3 - \frac{3}{5}\,x^2 - 2\,x + \frac{1}{3}$. **37.** $2\,x^2 + xy - 3\,y^2$. **39.** $(a + b)^3$. **41.** $\frac{1}{9}\,x^2 - 3\,y$. **43.** $ab + c$. **45.** $a^2 + b^2 + c^2 + d^2 - 2\,(ab - ac + ad - bd + bc + cd)$.

Article 12

1. $\frac{x}{2\,a}\left(x - \frac{y}{4\,a^2} - \frac{1}{2\,a}\right)$. **3.** $(x - 1)\,(3\,x - 1)$. **5.** $(3\,x - y)\,(2\,x + 7\,y)$. **7.** $x\,(2\,x - 3\,y)\,(4\,x^2 + 6\,xy + 9\,y^2)$. **9.** $(x + 2)\,(x - 2)\,(x + 3)\,(x - 3)$. **11.** $(x - 11)\,(x + 10)$. **13.** $(x - 9\,a^2)\,(x - a^2)$. **15.** $(xy - 5\,z)\,(xy + 2\,z)$. **17.** $(x - 1)\,(x - 8)\,(x + 8)$. **19.** $(x + 1)\,(x^2 - x + 1)\,(x - 1)\,(x^2 + x + 1)$. **21.** $-3\,xy\,(x + y)$. **23.** $(ac + b)\,(ac + d)$. **25.** $xy\,(x + y)\,(x - y)^2$. **27.** $(x^2y - 22)\,(x^2y + 5)$. **29.** $(x + 2)\,(x^2 + 7\,x + 2)$.

Article 15

1. $3\,(x + 1)$. **3.** $4\,(x^2 + y^2)$. **5.** $ax\,(a - x)^2$. **7.** $3\,a\,(2\,a + 3\,b - 4\,c)$. **9.** $(2\,x - 3\,y)$. **11.** $(3\,x - 2\,a)$. **13.** $(x^2 + 7)$. **15.** $(5\,x^2 - 1)$. **17.** $(x + y)$. **19.** $(a^2 - ab + b^2)$. **21.** $24\,a^2bx^2y^3$. **23.** $(a + b)\,(a - b)^2$. **25.** $(x - 4)\,(x + 1)\,(x + 3)$. **27.** $(3\,x - 2)\,(2\,x + 3)\,(2\,x - 3)$. **29.** $(3\,x - 2\,a)\,(4\,x - 3\,a)\,(3\,x + 4\,a)$. **31.** $(m + n)\,(m - n)\,(m + 2\,n)\,(m - 2\,n)$. **33.** $(x + 1)\,(x + 2)\,(x + 3)$. **35.** $(x - 1)\,(x + 1)\,(x + 2)\,(x + 3)$. **37.** $(x - 1)\,(x + 1)\,(x^2 + 1)\,(x^2 - x + 1)$. **39.** $(a - b)\,(a + b)\,(a^2 + ab + b^2)\,(a^2 - ab + b^2)\,(a^6 + a^3b^3 + b^6)$.

Article 19

1. $4ax$. **3.** $\frac{a^2-b^2}{x-y}$. **5.** $\frac{x^8+x^6y^2+x^4y^4+x^2y^6+y^8}{x^4y^4}$. **7.** 1. **9.** $\frac{x^2+x+1}{x^2+1}$. **11.** $\frac{1}{3(x-y)}$. **13.** $\frac{x(a+2x)}{a^2}$. **15.** $-\frac{v^2}{u^2}$. **17.** x^3+y^3. **19.** $\frac{4}{4x-y+3}$. **21.** 0. **23.** $\frac{2}{(x-1)(x-2)(x-3)}$. **25.** $\frac{-(3x^4-2x^3+3x-4)}{2(x+1)(x^2+1)(x^2-x+1)}$. **27.** $\frac{x^3}{a^3}-\frac{x^2}{a^2}+\frac{a^2}{x^2}-\frac{a^3}{x^3}$. **29.** $\frac{7x}{11y}\left(1-\frac{343x^3}{1331y^3}\right)$. **31.** $\frac{(5a^3-9c^3)(45c^3-49b^3)(9c^2-5a^3)}{14{,}175\,a^3c^6}$.

Article 21

5. $\frac{a^4b^6}{c^8d^{10}}$. **7.** $a^{12}b^9$. **9.** $\frac{m^{19}n^5q^6}{p^6}$.

Article 33

1. $\frac{3}{5}abc$. **3.** $\frac{(b^{2y}-a^{3x})^3}{a^{9x}b^{6y}}$. **5.** $\frac{y^2}{b^{\frac{8}{5}}x^2}$. **7.** $\frac{1}{x^{3n}a^{4n}b^{6n}}$. **9.** $\sqrt[6]{27}$, $\sqrt[6]{4}$. **11.** $\sqrt[12]{16}$, $\sqrt[12]{27}$. **13.** $\sqrt[12]{5^6}$, $\sqrt[12]{16}$, $\sqrt[12]{27}$. **15.** $\sqrt[12]{\frac{1}{16}}$, $\sqrt[12]{\frac{125}{512}}$, $\sqrt[12]{\frac{729}{15625}}$. **17.** $\sqrt[14]{\frac{25}{49}}$, $\sqrt[14]{\frac{2187}{128}}$, $\sqrt[14]{\frac{2}{11}}$. **19.** $\sqrt[6]{a^3}$, $\sqrt[6]{b^2}$, $\sqrt[6]{c}$. **21.** $\sqrt[12]{a^8}$, $\sqrt[12]{a^9}$, $\sqrt[12]{a^{10}}$. **23.** $\sqrt[30]{\frac{m^{10}}{n^{15}}}$, $\sqrt[30]{\frac{1}{y^{24}}}$, $\sqrt[30]{\frac{n^{10}}{y^{20}}}$. **25.** $\sqrt[mnp]{\frac{1}{x^{2np}}}$, $\sqrt[mnp]{x^{mp}}$, $\sqrt[mnp]{\frac{1}{x^{mn}}}$. **27.** $3\sqrt[3]{4}$. **29.** $9\sqrt{2}$. **31.** $3\frac{1}{2}\sqrt[3]{5}$. **33.** $(3+b-a)\sqrt{a}$. **35.** $a\sqrt{x}$. **37.** $1+\sqrt{3}$. **39.** $\frac{1}{2}(\sqrt{2}+\sqrt{6})$. **41.** $\frac{1}{2}(\sqrt{6}-\sqrt{14})$. **43.** $\sqrt{2}+\sqrt{5}$. **45.** $\sqrt{10}-\sqrt{3}$. **47.** 1. **49.** $6\sqrt{2}-3\sqrt{15}+8\sqrt{3}-6\sqrt{10}$. **51.** $8-8\sqrt[3]{12}+\sqrt[3]{18}$. **53.** $\sqrt{m^2-n}$. **55.** a. **57.** $\sqrt[15]{x^{32}y^{22}}$. **59.** 2. **61.** 4. **63.** 3. **65.** 3. **67.** $\sqrt[35]{\frac{a^{28}}{32}}$. **69.** $\frac{z}{3\sqrt{2}}$. **71.** $m^{\frac{10}{3}}$. **73.** $a^{\frac{25}{4}}$. **75.** $a^{\frac{8}{3}}$. **77.** $a^{\frac{25}{2}}$. **79.** $(x+y)^{\frac{15}{4}}$. **81.** $\frac{\sqrt{2}}{2}$. **83.** $a^{\frac{1}{3}}$. **85.** $\frac{a(1+\sqrt{a})}{1-a}$. **87.** $\frac{4a^3+12\sqrt{a^3}+9}{4a^3-9}$. **89.** $\frac{11+2\sqrt{14}}{5}$. **91.** $\frac{a\sqrt{b}-c\sqrt{d}}{a^2b-c^2d}$. **93.** $a^{\frac{5}{3}}$. **95.** $2a^{-2}-7a^{\frac{3}{2}}+6a^{-\frac{5}{4}}+7a^{-\frac{1}{2}}-11a^{-\frac{3}{4}}-2a^{-\frac{1}{3}}+7a^{-\frac{1}{4}}-6$. **97.** $2a^{\frac{5}{x}}-5a^{\frac{4}{x}}+10a^{\frac{3}{x}}-7a^{\frac{2}{x}}+6a^{\frac{1}{x}}$. **99.** $4a^{-2}b^{-\frac{4}{3}}-12a^{-\frac{8}{5}}b^{-\frac{22}{15}}+9a^{-\frac{6}{5}}b^{-\frac{8}{5}}$. **101.** $x^{\frac{3}{2}}-3xy^{\frac{2}{3}}+3x^{\frac{1}{2}}y^{\frac{4}{3}}-y^2$. **103.** $m^{-\frac{1}{2}}(1+4m^{-\frac{3}{2}}+6m^{-3}+4m^{-\frac{9}{2}}+m^{-6})$. **105.** $a^{\frac{4}{3}}+a^{\frac{3}{2}}+a^{\frac{8}{5}}+2(a^{\frac{17}{12}}-a^{\frac{22}{15}}-a^{\frac{31}{20}})$. **107.** $a^{\frac{1}{2}}+4b^{\frac{2}{3}}+9c+16d^{\frac{2}{5}}+2(-2a^{\frac{1}{4}}b^{\frac{1}{3}}+3a^{\frac{1}{4}}c^{\frac{1}{2}}-4a^{\frac{1}{4}}d^{\frac{1}{5}}-6b^{\frac{1}{3}}c^{\frac{1}{2}}+8b^{\frac{1}{3}}d^{\frac{1}{5}}-12c^{\frac{1}{2}}d^{\frac{1}{5}})$. **111.** $x^{\frac{4}{5}}-x^{\frac{3}{5}}+x^{\frac{2}{5}}-x^{\frac{1}{5}}+1$. **113.** $a^{\frac{1}{9}}+a^{\frac{1}{18}}b^{\frac{1}{18}}+b^{\frac{1}{9}}$. **115.** $a-\sqrt{a}$. **117.** $3\sqrt{-1}$; $5\sqrt{-1}$; $9\sqrt{-1}$. **119.** $3\sqrt{i}$.

121. $2\sqrt[4]{i}$. **123.** $9\sqrt{-1}$. **125.** $m\sqrt{-1}$. **127.** $47 - i$. **129.** $4i\sqrt{6} - 2$. **131.** -1. **133.** i. **135.** $\frac{a^2 - x}{a^2 + x} + i\frac{2a\sqrt{x}}{a^2 + x}$. **137.** $-\frac{1}{2}$. **139.** $2x = 3$. **141.** $ax + b = c^4$. **143.** $4x = 5$. **145.** $x = 5$. **147.** $x = 10$. **149.** $x = 4$.

Article 38

1. 0. **3.** $-3, -4, -6, -7$. **5.** $-3, -4$. **7.** 2, a. **9.** 7, .3. **13.** $(p+q)\left(1 + \frac{1}{pq}\right)$.

Article 41

3. 2.9196, 0.9196, 9.9196 − 10, 8.9196 − 10. **5.** 3.667. **7.** 1.655; 11.695. **9.** 52.22; 29.34. **11.** 0.1829. **13.** $\log\frac{ac}{bd}$. **15.** $\log\frac{\sqrt{u}\sqrt[3]{v}}{\sqrt[4]{w}}$. **17.** $\log\frac{a^3\sqrt[6]{ax+b}}{x+y}$.

Article 46

1. 0.975. **3.** 88.444. **5.** 0.99965.

Article 51

1. $x = \frac{5a - 3b}{2}$. **3.** $x = \frac{c}{b}$. **5.** $\frac{m-p}{q-n}$. **7.** ∞. **9.** $\frac{b}{b-a-1}$. **11.** $\frac{mn}{m+n-a}$; ∞. **13.** 1. **15.** $\frac{1}{4}$.

Article 60

1. 6. **3.** $\frac{ma}{m-1}$. **5.** $\frac{a}{n-m}$. **7.** 3 to 1. **9.** $1\frac{7}{8}$ days. **11.** $\frac{abc}{ab + ac + bc}$ days. **13.** $5\frac{5}{11}$ min. past 10; $21\frac{9}{11}$ min. past 10. **15.** $1\frac{1}{2}$ hrs.

Article 64

1. 3, 5. **3.** 32, − 17. **5.** 9, 8. **7.** 2, 3. **9.** Inconsistent. **11.** 0, 4. **13.** 1, − 1. **15.** Dependent. **17.** 6, 12. **19.** 12, 5.

Article 69

1. 6, 12. **3.** 6, 12. **5.** 9, 7. **7.** 4, 3. **9.** $x = \frac{1007.28}{1.0163725}$; $y = \frac{92.33}{1.0163725}$. **11.** Not independent. **13.** 5, 6. **15.** $\frac{1}{2}, \frac{1}{4}$. **17.** 4, 7. **19.** 7, $\frac{6}{7}$. **21.** $\frac{5a - 5b}{2}$, $\frac{a+b}{2}$. **23.** $\frac{abcg}{bg - af}$, $\frac{abcf}{bg - af}$. **25.** $x = \frac{nqrt + npsv}{mqr + ps}$; $y = \frac{qst - msqv}{ps + mqr}$. **27.** $\frac{m^2q}{mq - np}$, $\frac{nmq}{mq - np}$. **29.** No solution. **31.** No solution. **33.** 20, 17, 5. **35.** 3, 2, 1. **37.** 3, 4, 5. **39.** $\frac{1}{4}, \frac{1}{2}, \infty$. **41.** 1, 2, 3, 4. **43.** 1, .8, .2, .6. **45.** $16\frac{1}{12}$ hrs., $7\frac{11}{12}$ hrs. **47.** \$4000; $4\frac{1}{2}$%. **49.** 36, 9. **51.** 89, 35. **53.** 13, 17, 20. **55.** 1, $1\frac{3}{4}$, $1\frac{1}{2}$. **57.** 2, 3, 6 hrs. **59.** \$9150, \$8600, \$7550.

Article 75

1. $2, -6$. **3.** $2, -8$. **5.** $-2, 7$. **7.** $3, -4\frac{2}{3}$. **9.** $5, 1\frac{1}{2}$. **11.** $\frac{7}{4}, -\frac{1}{10}$. **13.** $-2\frac{1}{2}, 5$. **15.** $3, -4\frac{5}{7}$. **17.** $2b, -b$. **19.** $2, c$.

Article 86

1. 6. **3.** 0 or 3. **5.** 1. **7.** 13. **9.** 4. **11.** $-\frac{\sqrt{1441}-29}{4}$. **13.** $\frac{2}{3}$. **15.** 3 or $-\frac{2}{3}$. **17.** $1\frac{4}{5}$. **19.** 15. **21.** $\pm i\sqrt{\frac{7}{5}}$. **23.** 4 or $-\frac{1}{4}$. **25.** 3. **27.** $\pm\sqrt{-mp}$. **29.** $\pm b\sqrt{a^2-b^2}$. **31.** $\pm ia$. **33.** $\pm m\sqrt{-6}$. **35.** $\frac{m(1 \pm 2\sqrt{a})}{a+1}$. **37.** $b \pm \sqrt{b^2-ab}$. **39.** $a\sqrt{\frac{a}{a-1}}$. **41.** ± 8 or $\pm\frac{5i}{2}\sqrt{\frac{5}{2}}$. **43.** $\frac{a-b}{2} \pm \frac{a+b}{2}\sqrt{c^2-4}$. **45.** $4\sqrt[5]{4}$ or -8. **47.** 4 or -9. **49.** 27 or 64. **51.** 0 or 9. **53.** 14, 16, 18; or, $-14, -16, -18$. **55.** 30×60. **57.** $\frac{1}{2}\sqrt{ab-A}$. **59.** $\frac{b-a \pm \sqrt{a^2+b^2-6ab}}{2}$. **61.** $\frac{5}{2}$. **63.** $\frac{10}{5n-1}$, $n < \frac{1}{5}$. **65.** 20; 60. **67.** $x > -1$ and < -9; -1 and -9; $x < -1$ and > -9. **71.** $\frac{5}{2}(\sqrt{17}-1)$; $\frac{5}{2}(\sqrt{17}+1)$.

Article 93

1. $\pm\frac{1}{2}\sqrt{2}$; $\pm\frac{1}{2}\sqrt{2}$. **3.** $\frac{1}{2}\sqrt{2}$; $-\frac{1}{2}\sqrt{2}$. **5.** $\frac{-6 \pm \sqrt{11}}{5}$; $\frac{3 \pm 2\sqrt{11}}{5}$. **7.** 0 or $\frac{72}{25}$; 3 or $\frac{21}{25}$. **9.** $\frac{6 \pm \sqrt{6}}{20}$; $\frac{-2 \pm 3\sqrt{6}}{20}$. **11.** $m = \pm 2$.

Article 95

1. $\pm\frac{6}{\sqrt{13}}$; $\mp\frac{6}{\sqrt{13}}$. **3.** $\frac{9}{\sqrt{13}}$; $\frac{-4}{\sqrt{13}}$. **5.** 0, 2; 1, 0. **7.** $\frac{192 \pm 3\sqrt{-1559}}{145}$; $\frac{-162 \pm 2\sqrt{-1559}}{145}$. **9.** $\frac{4 \pm \sqrt{-23}}{13}$; $\frac{-9 \pm \sqrt{-23}}{13}$. **11.** $\pm\frac{1}{3}\sqrt{5}$.

Article 97

1. 0, 1; 0, 1. **3.** $\frac{65 \pm \sqrt{129}}{32}$; $\frac{1 \pm \sqrt{129}}{16}$. **5.** $\frac{7 \pm 4\sqrt{-2}}{9}$; $\frac{2 \mp 4\sqrt{-2}}{3}$. **7.** $-1 \pm \sqrt{5}$; $\frac{3 \mp \sqrt{5}}{2}$. **9.** $-4 \pm 2\sqrt{3}$; $-7 \pm 4\sqrt{3}$. **11.** -1.

Article 99

1. $1, -\frac{5}{4};\ 0, -\frac{3}{4}$. **3.** $\frac{1}{2}\sqrt{5};\ \frac{1}{2}$. **5.** $\frac{18 \pm 3\sqrt{-34}}{35};\ \frac{-2 \mp 12\sqrt{-34}}{35}$. **7.** $\frac{54 \pm \sqrt{66}}{25},\ \frac{-12 \mp 3\sqrt{66}}{25}$. **9.** $\pm\frac{12}{7}\sqrt{-7},\ \pm\frac{6}{7}\sqrt{-7}$. **11.** $\pm\sqrt{5}$.

Article 105

1. $x = \pm\frac{1}{2}\sqrt{2},\ \pm\frac{2}{5}\sqrt{5},\quad y = \pm\frac{1}{2}\sqrt{2},\ \mp\frac{1}{5}\sqrt{5}$. **3.** $x = 0,\ 3, \pm\frac{6}{13}\sqrt{13};\ y = 2,\ 0,\ \mp\frac{6}{13}\sqrt{13}$. **5.** $x = 0,\ 9,\ \frac{64}{9};\ y = 0, -6,\ \frac{16}{3}$.

Article 106

1. $\pm\frac{1}{2}\sqrt{29 \pm \sqrt{41}},\ \pm\frac{1}{2}\sqrt{7 \mp \sqrt{41}}$. **3.** $\pm\frac{9}{10}\sqrt{-5};\ \pm\sqrt{-\frac{29}{5}}$. **5.** $\pm\sqrt{3},\ \pm\frac{5}{19}\sqrt{57},\ 0,\ \mp\frac{2}{19}\sqrt{57}$.

Article 107

1. $\pm\sqrt{3};\ \pm 1$. **3.** $\pm\frac{\sqrt{5}}{3};\ \pm\frac{4}{3}$. **5.** $\pm\frac{\sqrt{65}}{3};\ \pm\frac{4i}{3}$.

Article 111

1. $x = \pm 25;\ y = \pm 6$. **3.** $\pm 5;\ \pm 4$. **5.** $\frac{1 \pm \sqrt{5}}{2};\ \frac{1 \mp \sqrt{5}}{2}$. **7.** $\frac{2}{3},\ \frac{1}{2};\ \frac{1}{2},\ \frac{2}{3}$. **9.** $7,\ \frac{499}{133};\ 8,\ 9\frac{232}{665}$. **11.** 13; 7. **13.** $\pm 13;\ \pm 7$; two solutions. **15.** $7, -\frac{6}{5};\ -3,\ 17\frac{1}{2}$. **17.** $37\frac{7}{11},\ 4;\ 43\frac{5}{11},\ 7$. **19.** 4, 5; 4, 3; two answers. **21.** $14\frac{51}{71},\ 5;\ 15\frac{43}{71},\ 2$. **23.** 8, 9; 9, 8. **25.** $\pm 2, \infty;\ \pm 1, \infty$. **27.** $\frac{\pm 1 \pm \sqrt{5}}{2};\ \frac{\mp 1 \pm \sqrt{5}}{2}$; four solutions. **29.** $3 \pm \sqrt{6},\ \frac{-1 \pm \sqrt{-11}}{2};\ 3 \mp \sqrt{6},\ \frac{-1 \mp \sqrt{-11}}{2}$. **31.** $-2, \infty;\ 0, \infty$. **33.** 7, 2; 2, 7. **35.** 0, 5; 5, 0. **37.** $5, -6;\ 11, -12$; four answers. **39.** $2,\ 3,\ -3 \pm \sqrt{3};\ 3,\ 2,\ -3 \mp \sqrt{3}$. **41.** $12, 3,\ -8 \pm 2\sqrt{7};\ 3, 12,\ -8 \mp 2\sqrt{7}$. **43.** $18,\ \frac{8}{3},\ \frac{-21 \pm \sqrt{249}}{2};\ 8,\ 54,\ \frac{-63 \mp 3\sqrt{249}}{2}$. **45.** $4, 3,\ \frac{7 \pm \sqrt{-295}}{2};\ 3, 4,\ \frac{7 \mp \sqrt{-295}}{2}$. **47.** $4, 7,\ \frac{11 \pm \sqrt{-735}}{2};\ 7,\ 4,\ \frac{11 \mp \sqrt{-735}}{2}$. **49.** 9, 7; 7, 9; two answers. **51.** $\pm 2,\ \pm\frac{9}{172}\sqrt{516};\ \pm 1 \mp \frac{31}{\sqrt{516}}$. **53.** $x = 7,\ -2,\ 7w,\ -2w,\ 7w^2,\ -2w^2;\ y = 2,\ -7,\ 2w,\ -7w,\ 2w^2,\ -7w^2;\ w \equiv \frac{-1 + \sqrt{-3}}{2}$. **55.** $m = 11,\ -9,\ 11w,\ -9w,\ 11w^2,\ -9w^2;\ n = 9,\ -11,\ 9w,\ -11w,\ 9w^2,\ -11w^2$.

57. $x = 243, 32$; $y = 64, 729$; two answers. **59.** $x = 3, -1, +1, -3$; $y = 1, -3, 3, -1$. **61.** $\pm\sqrt{1 \pm \frac{1}{3}\sqrt{3}}$; $\pm\sqrt{1 \mp \frac{1}{3}\sqrt{3}}$. Use both upper or both lower signs under radicals; outside of radicals use all combinations. **63.** $\frac{\pm\sqrt{m^2+4n^2}+m}{2}$; $\frac{\pm\sqrt{m^2+4n^2}-m}{2}$; two solutions. **65.** $\frac{b^2 \pm a\sqrt{2b^2-a^2}}{2}$; $\frac{b^2 \mp a\sqrt{2b^2-a^2}}{2}$; two solutions. **67.** $\frac{m}{2a}\left(\pm\sqrt{\frac{n+abm}{n-3abm}}+1\right)$; $\frac{m}{2a}\left(\pm\sqrt{\frac{n+abm}{n-3abm}}-1\right)$; two solutions. **69.** $x = \pm\sqrt{-5}+1$, $y = \pm\sqrt{-5}-1$; $x = \pm\sqrt{-1}+1$, $y = \pm\sqrt{-1}-1$; four solutions. **71.** $u = \pm\frac{1}{2}\sqrt{\pm 32\sqrt{2}-27}+1$; $v = \pm\frac{1}{2}\sqrt{\pm 32\sqrt{2}-27}-1$; four solutions. Use all possible combinations of signs in u and in v. **73.** $\pm ab\sqrt{2a^2-b^2}-ab^2$; $\pm ab\sqrt{2a^2-b^2}+ab^2$; two solutions. **75.** $\frac{p}{\sqrt[3]{p^2+q^2}}$; $\frac{q}{\sqrt[3]{p^2+q^2}}$. **77.** $5, 3, 4 \pm\sqrt{-33}$; $3, 5, 4 \mp\sqrt{-33}$. **79.** $x = \pm\sqrt{-3}$, $y = \mp\sqrt{-3}$, $z = 2$; two solutions. **81.** $x = 2$ or ∞; $y = -\frac{1}{2}$ or -1; $z = 1$ or 0. **83.** $x = \frac{1}{2q}\left(pq - r \pm \sqrt{(pq-r)^2 - 4q^3}\right)$; $y = \frac{1}{2q}\left(pq - r \mp \sqrt{(pq-r)^2-4q^3}\right)$; $z = \frac{r}{q}$; two solutions. **85.** $\pm\frac{19}{4}, \mp\frac{11}{4}, \pm\frac{13}{4}$; take all upper or all lower signs. **87.** $x = \frac{1}{4}\left(a-b-c-2 \pm \sqrt{(2+b+c-a)^2+4a(2+c)}\right)$; $y = \frac{b}{1+z}$; $z = \frac{c}{1+x}$. **89.** $x = \frac{4}{3}$ or $\frac{2}{3}$; $y = \pm\frac{1}{4}\sqrt{-\frac{7}{3}}$, or $\pm\frac{3}{4}$; $z = \pm\frac{1}{2}\sqrt{-\frac{5}{3}}$, or $\pm\frac{1}{2}$.

PROBLEMS

1. 8, 6. **3.** 48, 36. **5.** $x = 15, -12$; $y = 11, -16$; two answers. **7.** $x = 19, -20$; $y = 17, -18$; four answers. **9.** 33, 56. **11.** 19, 23. **13.** 28, 20 ft. sec. **15.** $13\frac{7}{41}$; 45 days. Assume each man's pay proportional to amount of work he does. **17.** 42. **19.** $\frac{9}{12}$. **21.** 3, 5 yds. **23.** $s_1 = 15.4$; 11.7 ft. sec.; $s_2 = 6.8$; 12.2 ft. sec.

Article 114

1. 3. **3.** -3. **5.** 0. **7.** -3. **9.** 1. **11.** -4. **13.** $-\frac{9}{4}$. **15.** $\frac{9}{2}$. **17.** $\frac{\log 3}{\log \frac{25}{3}}$. **19.** $\frac{\log a^2b}{\log a^{-3}b^2}$. **21.** 1, -3. **23.** $-\frac{16}{5}$. **25.** -6.

Article 122

25. 1. **27.** $\sqrt{600{,}000}$. **29.** $6\frac{3}{4}$ in.

Article 148

1. $\frac{\pi}{4}+n\pi$; $2n\pi - \frac{\pi}{6}$, $(2n+1)\pi + \frac{\pi}{6}$; $\pm\frac{\pi}{3} \pm 2n\pi$; $2n\pi$. **3.** $2n\pi - 41°\,48'$, $(2n+1)\pi + 41°\,48'$; $(2n+1)\pi \pm 70°\,32'$; $63°\,26' + n\pi$; $2n\pi + 11°\,32'$; $(2n+1)\pi - 11°\,32'$. **5.** $68°\,12' + n\pi$; $2n\pi - 16°\,35'$; $(2n+1)\pi + 16°\,35'$; $2n\pi \pm 5°\,44'$.

Article 150

	sin	cos	tan	csc	sec	cot
1.	$-\frac{1}{2}$	$\pm\frac{1}{2}\sqrt{3}$	$\pm\frac{1}{\sqrt{3}}$	-2	$\pm\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
3.	$\pm\frac{4}{5}$	$\pm\frac{3}{5}$	$\frac{4}{3}$	$\pm\frac{5}{4}$	$\pm\frac{5}{3}$	$\frac{3}{4}$
5.	$\pm\frac{1}{2}$	$\pm\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	± 2	$\pm\frac{2}{\sqrt{3}}$	$\sqrt{3}$
7.	$\pm\frac{9}{41}$	$-\frac{40}{41}$	$\pm\frac{9}{40}$	$\pm\frac{41}{9}$	$-\frac{41}{40}$	$\pm\frac{40}{9}$
9.	$.6$	$\pm\frac{4}{5}$	$\pm\frac{3}{4}$	$\frac{5}{3}$	$\pm\frac{5}{4}$	$\pm\frac{4}{3}$
11.	$-\frac{n}{m}$	$\pm\frac{\sqrt{m^2-n^2}}{m}$	$\pm\frac{n}{\sqrt{m^2-n^2}}$	$-\frac{m}{n}$	$\pm\frac{m}{\sqrt{m^2-n^2}}$	$\pm\frac{\sqrt{m^2-n^2}}{n}$
13.	h	$\pm\sqrt{1-h^2}$	$\pm\frac{h}{\sqrt{1-h^2}}$	$\frac{1}{h}$	$\pm\frac{1}{\sqrt{1-h^2}}$	$\pm\frac{\sqrt{1-h^2}}{h}$
15.	$\pm\frac{a^2-b^2}{a^2+b^2}$	$\frac{2ab}{a^2+b^2}$	$\pm\frac{a^2-b^2}{2ab}$	$\pm\frac{a^2+b^2}{a^2-b^2}$	$\frac{a^2+b^2}{2ab}$	$\pm\frac{2ab}{a^2-b^2}$

Article 151

1. $\pm\frac{\tan x}{\sqrt{1+\tan^2 x}}$. **3.** $\frac{2\sqrt{\csc^2 x-1}}{\csc^2 x}$. **5.** $\cos\theta \pm \sqrt{1-\cos^2\theta}$.

Article 159

1. 1; 0. **3.** $\frac{156}{205}$; $\frac{133}{205}$. **5.** $\pm\frac{\sqrt{5}\pm 4\sqrt{2}}{9}$; $\pm\frac{\sqrt{5}\mp 4\sqrt{2}}{9}$. **7.** $\pm\frac{\sqrt{7}}{111}$ $[(12 \pm 63) \pm (18 \pm 14)\sqrt{3}]$. **9.** $\frac{10437}{13325}$. **11.** $\frac{16}{65}$. **13.** $\pm\frac{2016}{4225}$, $\pm\frac{1056}{4225}$, $\pm\frac{3696}{4225}$; $\frac{253}{204}$. **15.** $\frac{a^2}{2}$. **17.** $\frac{1}{2}$. **19.** $\sin 202\frac{1}{2}° = \frac{1}{2}\sqrt{2-\sqrt{2}}$; $\cos 202\frac{1}{2}° = -\frac{1}{2}\sqrt{2+\sqrt{2}}$; $\tan 202\frac{1}{2}° = \sqrt{3-2\sqrt{2}}$, $\sin 7\frac{1}{2}° = \frac{1}{2}\sqrt{2\sqrt{2}-\sqrt{3}-1}$; $\cos 7\frac{1}{2}° = \frac{1}{2}\sqrt{2\sqrt{2}+\sqrt{3}+1}$; $\tan 7\frac{1}{2}° = \sqrt{15+8\sqrt{3}-10\sqrt{2}-6\sqrt{6}}$.

Article 160

1. 4° 40′, 3° 20′. **3.** 8; 5.

Article 166

1. $2n\pi \pm 60°$. **3.** $(n+\frac{1}{4})\pi$. **5.** $n\pi + 45°$; $n\pi + 71° 34'$. **7.** $2n\pi + 36° 52'$. **9.** $n\pi$. **11.** $n\pi$; $n\pi \pm \frac{\pi}{4}$. **13.** $n\pi$; $n\pi \pm \frac{\pi}{6}$. **15.** $(2n+1)\frac{\pi}{8}$. **17.** $\frac{4n\pm 1}{2(p\pm q)}\pi$. **19.** $\frac{2n\pi}{r-s}$; $\frac{n\pi}{r+s}$. **21.** $n\pi$; $135° + n\pi$. **23.** $2n\pi - 60°$; $2n\pi - 120°$. **25.** $2n\pi + 30°$; $(2n+1)\pi - 30°$. **27.** $\frac{n\pi}{2}$; $n\pi + \frac{\pi}{4}$. **29.** $\frac{n\pi}{3}$; $(2n+1)\frac{\pi}{2} \pm 30°$.

Article 168

1. $r = \pm 5, \theta = \tan^{-1}\frac{4}{3}$. **3.** $r = \pm 41, \theta = \tan^{-1}\frac{40}{9}$. **5.** $r = \pm\frac{\sqrt{2}}{5}, \theta = \tan^{-1} 1$. **7.** $r = \pm 3\sqrt{5}, \theta = \tan^{-1}(-3)$. **9.** $r = 5\sqrt{2}, \phi = \tan^{-1}\frac{4}{3}, \theta = \tan^{-1} 1$. **11.** $x^2 + y^2 = r^2$. **13.** $\frac{x^{\frac{2}{3}}}{a^2} + \frac{y^{\frac{2}{3}}}{b^2} = 1$. **15.** $x^2 + y^2 + z^2 = 1$.

Article 179

1. Area = 4828, $A = 97° 48'$, $B = 18° 21'.8$, $C = 63° 50'.2$. **3.** Area = 1445.7, $A = 34° 24'$, $B = 73° 15'$, $C = 72° 21'$. **5.** $b = 290.9$, $c = 289.0$, $B = 72° 6'$. **7.** $b = 5340$, $c = 6535$, $A = 81° 52'$. **9.** $a = 9548$, $c = 10804$, $C = 105° 59'$. **11.** No solution. **13.** $c = 3120$, $c' = 402.2$, $B = 26° 52'$, $B' = 153° 8'$, $C = 131° 47'$, $C' = 5° 31'$. **15.** $b = .5458$, $b' = .1814$, $A = 39° 37'$, $A' = 140° 23'$, $B = 117° 51'$, $B' = 17° 5'$. **17.** $c = .7105$, $A = 76° 20'$, $B = 44° 53'$, Area = .2024. **19.** $a = 13.534$, $B = 15° 9'.4$, $C = 131° 19'.6$, Area = 32.564. **21.** $A = 149° 49'$, $B = 3° 2'$, $C = 27° 9'$. **23.** $B = 51° 9'$, $B' = 128° 51'$, $C = 87° 38'$, $C' = 9° 56'$, $c = 116.82$, $c' = 20.172$. **25.** $b = 71760$, $B = 146° 43'$, $C = 14° 4'$. **27.** $A = 57° 53'$, $B = 70° 17'$, $C = 51° 50'$. **29.** $c = 38088$, $B = 48° 34'.7$, $C = 49° 38'.3$. **31.** $A = 18° 12'$, $B = 135° 51'$, $C = 25° 57'$. **33.** $c = 748.1$, $A = 42° 51'$, $B = 64° 9'$. **35.** $b = .000331$, $B = 83° 33'$, $C = 32° 36'$. **37.** $c = 2406$, $c' = 227.6$, $B = 31° 58'$, $B' = 148° 2'$. $C = 120° 44'$, $C' = 4° 40'$. **39.** $c = 369.27$, $A = 39° 39'.6$, $C = 90°$. **63.** 7; $\sqrt{129}$; $20\sqrt{3}$. **65.** 6824. **67.** 45°, 60°, 75°; 612.5 ft.; 683 ft. **69.** 698.3 ft. **71.** 121 ft.; 390 ft. **73.** 1145 ft. **75.** 8640 ft. **77.** 62.00 ft. **79.** 969.2 ft. **81.** 19955 m. **83.** 59.1; 513. **85.** 25, $33\frac{1}{3}$, $41\frac{2}{3}$. **87.** $\frac{45\sqrt{5}}{4}$. **89.** 18.76 chains; 7.578 acres. **91.** 3.620 acres, south of dividing line. **93.** 10.802 chains east of A. **95.** $i = \tan^{-1}\frac{8}{19}$. **97.** 20° 7′. **99.** 12° 32′.

Article 183

1. 55; 403. **3.** 14; 200. **5.** 28; 364. **7.** $p - \frac{19}{2}q$; $20p - 95q$. **9.** $l = 150$; $d = 3$. **11.** $a = 9$; $d = 2$. **13.** $a = 18$; $d = 5$. **15.** $a = 17$; $l = 97$. **17.** $a = \frac{1}{3}$; $l = \frac{50}{3}$. **19.** $n = 16$, $l = 69$. **21.** $n = 14$; $a = 12$. **23.** $n = 103$, $a = 1281$. **25.** 8925. **27.** 10 sec. **29.** 29700 ft.

Article 187

1. $l = 256$; $S = 508$. **3.** $l = 4096$; $S = 5461$. **5.** $l = \frac{1}{262144}$; $S = \frac{349525}{262144}$. **7.** $l = a(1 + x)^7$; $S = \frac{a(1 + x)^8 - a}{x}$. **9.** ± 48; 288, ± 1728. **11.** ± 12, 4, $\pm\frac{4}{3}$, $\frac{4}{9}$, $\pm\frac{4}{27}$. **13.** 12, 3, $\frac{3}{4}$, $\frac{3}{16}$. **15.** $a = 2$, $S = 254$. **17.** $a = 6$, $S = \frac{242}{27}$. **19.** $n = 6$, $S = 126$. **21.** $r = 3$, $n = 7$. **23.** $r = \frac{1}{2}$, $n = 6$. **25.** $n = 9$, $l = 19683$. **27.** $a = 5$, $l = 320$.

Article 189

1. $3\frac{2}{3}$. **3.** $\frac{25}{2}$. **5.** $\frac{2}{3}$. **7.** 8 sec.

Article 191

1. $a = 115$ or 1; $d = -10$ or $+2$. **3.** $a = -11$ or $\frac{755}{7}$; $d = 4$ or $-\frac{100}{7}$. **5.** First number $\frac{11}{4}$; com. diff. $\frac{1}{12}\sqrt{2989}$ or $\frac{1}{12}\sqrt{-1779}$. **7.** Middle number $= b$; com. diff. $= \pm\sqrt{\frac{1}{8}(5b^2) \pm \sqrt{9b^4 + \frac{16a}{b}}}$. **9.** 55°, 60°, 65°. **11.** a, $ar^{\frac{1}{2}}$, ar, $ar^{\frac{3}{2}}$, ... **13.** $\pm\frac{5}{2}$, ± 10, ± 40, ± 160. **15.** 10.11 inches. **17.** $\$1845 \times 10^{10}$. **19.** $2a$; $4a\sqrt{3}$.

Article 194

1. \$2975+. **3.** \$1489+. **5.** 20. **7.** \$497.80. **9.** $\frac{A}{r(1+r)^{m-1}}$.

Article 203

1. Convergent. **3.** Conv. if $|x| < 1$. Div. if $|x| \geqq 1$. **5.** Conv. if $|x| < \frac{\pi}{4}$. **7.** Conv. if $1 < x < 10$. **9.** Convergent. **11.** Convergent for all values of x. **13.** Conv. for all values of x. **15.** Conv. when $-1 < x \leqq 1$.

Article 205

1. .41. **3.** 1.261. **5.** .0589+. **7.** .0053+.

Article 209

1. $\frac{3}{8}x^2$. **3.** $3x^2 - 1$. **5.** $\pm\frac{3}{2\sqrt{-x}}$. **7.** $\pm\frac{x}{\sqrt{x^2-1}}$. **9.** $\pm\frac{2x}{\sqrt{x^2-1}}$.

Article 214

1. $12x^3 + 15x^2$. **3.** $\frac{1}{4x^{\frac{1}{2}}} + \frac{1}{9x^{\frac{2}{3}}}$. **5.** $-\frac{1}{2x^{\frac{3}{2}}} - \frac{1}{3x^{\frac{4}{3}}}$. **7.** $2xe^{x^2}$. **9.** $-\sin x + \sec x \tan x$. **11.** $\frac{6x}{3x^2-1}$. **13.** $-\sin x \tan x + \cos x \log \cos x$. **15.** $\sec^2 x$. **17.** $\sec x \csc x$.

Article 216

1. $3x^2 + 2x$. **3.** $\cos^2 x - \sin^2 x$. **5.** e^x. **7.** $12l$, where l = length of edge. **9.** Area of base. **11.** $\frac{\text{Perimeter}}{2\pi}$. **13.** $6t^2 + 3$; 603; 9; 3. **15.** $\frac{da}{db} = \frac{b - c\cos A}{a}$, $\frac{da}{dc} = \frac{c - b\cos A}{a}$, $\frac{da}{dA} = \frac{bc\sin A}{a}$.

Article 222

1. $x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$. **3.** $x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \cdots$. **5.** $1 + 2x + 2x^2 + \frac{4x^3}{3} + \cdots$. **7.** $2 + \frac{2x^2}{\lfloor 2} + \frac{2x^4}{\lfloor 4} + \frac{2x^6}{\lfloor 6} + \cdots$. **9.** $1 - \frac{x}{a} + \frac{x^2}{2a^2} - \frac{x^3}{6a^3} + \cdots$. **11.** $1 + x - \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \cdots$. **13.** $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$. **15.** $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5x^3}{16} + \cdots$. **17.** $1 + 2x + 2x^2 + 2x^3 + \cdots$. **19.** $3^{\frac{1}{3}} - \frac{x}{2\cdot 3^{\frac{5}{3}}} - \frac{x^2}{4\cdot 3^{\frac{11}{3}}} - \frac{5x^3}{8\cdot 3^{\frac{20}{3}}} + \cdots$. **21.** $2^{\frac{4}{7}} - \frac{4x^3}{7\cdot 2^{\frac{3}{7}}} - \frac{6x^6}{49\cdot 2^{\frac{10}{7}}} - \frac{20x^9}{343\cdot 2^{\frac{17}{7}}} + \cdots$. **23.** $\frac{1}{6^{\frac{3}{2}}}\left[\sqrt{27x^3} + 6\sqrt{3x} + \frac{6}{\sqrt{3x}} - \frac{4}{\sqrt{27x^3}} + \cdots\right]$. **25.** $\frac{1}{(16a)^{\frac{1}{4}}} - \frac{4x^{\frac{2}{3}}}{(16a)^{\frac{7}{12}}} + \frac{14x^{\frac{4}{3}}}{(16a)^{\frac{5}{6}}} - \frac{140x^2}{(16a)^{\frac{13}{12}}} + \cdots$.

Article 229

1. $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. **3.** $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. **5.** $n\left(a + \frac{n-1}{2}d\right)$.

Article 231

3. .0314; .0204. **5.** $5^h\,16^m\,05^s.59$; $18^\circ\,48'\,10''.1$.

Article 232

1. Sixth entry should be .364. **3.** Sixth entry should be 3′ 30″.

Article 234

1. $a_n = a_{n-2} - a_{n-1}$; $n > 1$. **3.** $1 + x^2 + x^3 + 2x^4 + \cdots$. **5.** $\frac{3}{2}x - \frac{1}{2}x^2 - \frac{1}{4}x^3 + \frac{1}{2}x^4 + \cdots$. **7.** $\frac{2}{3x} - \frac{17}{9} + \frac{83x}{27} - \frac{383x^2}{81} + \cdots$.

Article 239

1. $\frac{3}{8(3x+1)} - \frac{1}{8(x+3)}$. **3.** $\frac{2}{x} - \frac{2}{x+2} + \frac{1}{x-2}$. **5.** $\frac{2\sqrt{3}+3}{6(x-2-\sqrt{3})} - \frac{2\sqrt{3}-3}{6(x-2+\sqrt{3})}$. **7.** $\frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}$. **9.** $\frac{3}{x} + \frac{5-3x}{x^2+4}$. **11.** $1 - \frac{1}{3x} + \frac{x-9}{3(x^2+3)}$. **13.** $-\frac{2}{x} + \frac{2}{x-2} - \frac{3}{(x-2)^2}$. **15.** $\frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)}$. **17.** $\frac{1}{2(x-1)} + \frac{1}{5(x-2)} + \frac{3}{10(x+3)}$. **19.** $\frac{4}{x-2} - \frac{1}{x-1}$. **21.** $\frac{1}{x+1} - \frac{2}{x+2} + \frac{2}{x-2}$.

Article 241

3. $x = \frac{13}{16}$, $y = \frac{1}{8}$, $z = \frac{5}{16}$. **5.** $x = \frac{10}{11}$, $y = 1$, $z = \frac{24}{11}$. **7.** Not independent.

Article 249

1. 0. **3.** 0. **5.** 8. **7.** 398. **9.** 832. **11.** $a_1b_2c_3d_4$. **33.** $u = \frac{49}{100}$, $v = -\frac{7}{5}$, $w = \frac{28}{25}$. **35.** Inconsistent.

Article 257

1. $\sqrt{2}$, $-45°$; 5, 36° 52′; $\sqrt{146}$, 114° 27′; 2, 90°; 2, 0°; 2, 0°; 6, 30°; 36, $-60°$; 4, 90°.

Article 259

3. ± 3; $\pm 3i$. **5.** $x_1 = 2$; $x_2 = 2(\cos 72° + i \sin 72°)$; $x_3 = 2(\cos 144° + i \sin 144°)$; etc. **7.** $x_1 = \sqrt{3}$; $x_2 = \dfrac{\sqrt{3} + 3i}{2}$; $x_3 = \dfrac{-\sqrt{3} + 3i}{2}$; etc.

Article 260

1. $3\cos^2\theta \sin\theta - \sin^3\theta$. **3.** $\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$. **5.** $6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$.

Article 263

1. 24. **3.** 240.

Article 264

1. 20. **3.** 120. **7.** 190.

Article 266

1. 1260. **3.** 360. **5.** ${}_6C_4 \times {}_{14}C_7 + {}_6C_5 \times {}_{14}C_6 + {}_6C_6 \times {}_{14}C_5 = 71500$. **7.** 73. **9.** 4; there will be three different throws. **11.** 36; there will be 21 different throws.

Article 268

1. $\frac{5}{16}$. **3.** $\frac{5}{42}$. **5.** $\frac{20}{21}$. **7.** $\frac{1}{256}$. **9.** $\frac{15}{256}$. **11.** $\frac{66}{195755}$.

Article 270

1. $\frac{4}{15}$. **3.** $\frac{4}{45}$; $\frac{1}{9}$. **5.** 6. **7.** $\frac{4}{19}$. **9.** $\frac{1}{2}$.

Article 275

1. $x^3 - 6x^2 + 11x - 6 = 0$. **3.** $x^4 - 2x^3 - 4x^2 + 8x = 0$. **5.** $6x^4 - 5x^3 - 5x^2 + 5x - 1 = 0$.

Article 280

1. -1, 2, 2. **3.** 3, 3, -2, -2. **5.** 3, 3, -1, -2. **7.** 1, 1, 1, -2. **9.** 3, 3, $\pm\frac{1}{3}$.

Article 286

1. $x^3 + 2x^2 - 4x - 8 = 0$. **3.** $x^3 - 12x - 14 = 0$. **5.** $x^2 + 2x + 1 = 0$. **7.** $x^2 - 2x - 2 = 0$. **9.** $x^3 + x^2 - 9 = 0$. **11.** $x^3 - 9x^2 + 24x - 16 = 0$. **13.** $x^4 + 6x^3 + x^2 - 24x + 16 = 0$. **15.** $h = 1$; $x^3 - 3x = 0$. **17.** $h = 1$; $x^3 - 9x - 7 = 0$. **23.** $\pm\sqrt{-1}, 2$. **25.** $2, \pm 2\sqrt{2}, -1 \pm \sqrt{-3}$.

Article 291

1. $2, 2, -1$. **3.** $1, -\frac{1}{2}, -\frac{1}{2}$. **5.** $3, 2 \pm 2\sqrt{3}$. **7.** $-1, -2, 3$. **9.** $1, -1, -1 \pm \sqrt{-2}$. **11.** $\frac{1}{2}(-1 \pm \sqrt{5}), \frac{1}{2}(5 \pm \sqrt{37})$. **13.** $2, -2, -2, -2$. **15.** $4, 2, -1 \pm \sqrt{-3}$.

Article 305

1. $A = 79°\ 30'.8$, $B = 46°\ 15'.3$, $C = 70°\ 55'.6$. **3.** $A = 130°\ 5'.4$, $B = 32°\ 26'.1$, $C = 36°\ 45'.8$. **5.** $a = 96°\ 24'.5$, $b = 68°\ 27'.4$, $c = 87°\ 31'.6$. **7.** $c = 50°\ 6'$, $A = 129°\ 58'$, $B = 34°\ 30'$. **9.** $a = 43°\ 18'$, $B = 28°\ 48'$, $C = 74°\ 22'$. **11.** $b = 78°\ 17'$, $c = 126°\ 46'$, $A = 96°\ 46'$. **13.** $a = 76°\ 25'$, $b = 58°\ 19'$, $C = 116°\ 31'$. **15.** $a = 124°\ 12'\ 31''$, $c = 97°\ 12'\ 25''$, $B = 51°\ 18'\ 11''$. **17.** $a = 58°\ 8'\ 19''$, $B = 98°\ 20'\ 0''$, $C = 63°\ 40'\ 0$. **19.** $b = 75°\ 29'$, $c = 108°\ 14'$, $C = 46°\ 52'$. **21.** No solution. **23.** $c = 84°\ 30'$, $B = 56°\ 20'$, $C = 97°\ 19'$. **25.** $B = 42°\ 37'\ 18''$, $137°\ 22'\ 42''$, $C = 160°\ 1'\ 24''$, $50°\ 18'\ 55''$, $c = 153°\ 38'\ 42''$; $90°\ 5'\ 41''$. **27.** $a = 64°\ 23'\ 20''$, $b = 99°\ 48'\ 50''$, $A = 65°\ 33'\ 10''$.

Article 306

1. N.Y. – S.F. 2568 mi. N.Y. – M.C. 2090 mi. S.F.–M.C. 1889 mi. Angles: N.Y. 48° 58′, S.F. 55° 48′, M.C. 82° 40′. Area: 2025300 sq. mi. **3.** $\lambda = 9^h\ 34^m\ 15^s$, $\phi = 22°\ 6'$ N; course, S 44° 28′ W.

Article 307

1. $A = \pm 92°\ 50'$; $t = \pm 5^h\ 4^m\ 12^s$. **3.** $h = 43°\ 27'$; $A = 70°\ 3'$. **5.** N 1° 33′.6 E or W; $t = \pm 5^h\ 55^m\ 54^s$. **7.** 9° 46′.4.

INDEX

APPENDIX A

THE GREEK ALPHABET

Letters.	Name.	Letters.	Name.	Letters.	Name.
A, α,	Alpha	I, ι,	Iota	P, ρ,	Rho
B, β,	Beta	K, κ,	Kappa	Σ, σ,	Sigma
Γ, γ,	Gamma	Λ, λ,	Lambda	T, τ,	Tau
Δ, δ,	Delta	M, μ,	Mu	Υ, υ,	Upsilon
E, ϵ,	Epsilon	N, ν,	Nu	Φ, ϕ,	Phi
Z, ζ,	Zeta	Ξ, ξ,	Xi	X, χ,	Chi
H, η,	Eta	O, o,	Omicron	Ψ, ψ,	Psi
Θ, θ, ϑ,	Theta	Π, π,	Pi	Ω, ω,	Omega

LIST OF FORMULAS

Factors of $a^n \pm b^n$, n being a positive integer **(9).**

$a^n - b^n$ is divisible by $(a - b)$ and by $(a + b)$ when n is even.

$a^n - b^n$ is divisible by $(a - b)$, not by $(a + b)$, when n is odd.

$a^n + b^n$ is divisible by $(a + b)$, not by $(a - b)$, when n is odd.

$a^n + b^n$ is not divisible by $(a + b)$ or by $(a - b)$ when n is even.

Special Cases.

$a^2 - b^2 = (a + b)(a - b)$. $a^2 + b^2$ has no real factors.

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$a^4 - b^4 = (a^2 + b^2)(a^2 - b^2)$. $a^4 + b^4$ has no real factors.

$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

Factor Theorem. — If $f(x)$ reduces to zero when $x = a$, $f(x)$ contains the factor $(x - a)$. **(11), (272).**

Exponents. (20) to (25).

$$a^0 = 1. \quad a^{-x} = \frac{1}{a^x}. \quad a^{\frac{1}{x}} = \sqrt[x]{a}.$$

$$a^x a^y = a^{x+y}. \quad a^x \div a^y = a^{x-y}.$$

$$(a^x)^y = a^{xy}. \quad (ab)^x = a^x b^x. \quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}.$$

Imaginary or Complex Numbers. (26.)

$$i \equiv \sqrt{-1}; \quad i^2 = -1; \quad i^3 = -i; \quad i^4 = +1, \text{ etc.}$$

$$\sqrt{-a} \equiv i\sqrt{a}. \quad a^2 + b^2 = (a + ib)(a - ib).$$

$$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Surds. — If $a + \sqrt{b} = c + \sqrt{d}$, where $\sqrt{b}$ and $\sqrt{d}$ are surds, then $a = c$ and $b = d$. (29.)

Logarithms. (37), (39), (226).

If $a^x = m$, then $x = \log_a m$.

$$\log_a mn = \log_a m + \log_a n. \qquad \log_a \frac{m}{n} = \log_a m - \log_a n.$$

$$\log_a m^p = p \log_a m. \qquad \log_a \sqrt[p]{m} = \frac{1}{p}\log_a m.$$

$$\log_a a = 1. \quad \log_a 1 = 0. \qquad \log_a 0 = -\infty, \text{ if } a > 1.$$

Change of Base. $\log_a m = \log_b m \times \log_a b.$

If $a = 10$ and $b = e$, then $\log_a b = \log_{10} e = M$. (Table V.)

Hence $\log_{10} m = M \log_e m.$

Binomial Theorem. (42), (220–1).

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{\lfloor 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{\lfloor 3}a^{n-3}b^3 + \cdots$$

$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}a^{n-r}b^r + \cdots.$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n(n-1)(n-2)}{\lfloor 3}x^3 + \cdots.$$

Quadratic Equation, $ax^2 + bx + c = 0$. **(74), (76), (78).**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Roots real and unequal if $b^2 - 4ac > 0$.
Roots real and equal if $b^2 - 4ac = 0$.
Roots imaginary if $b^2 - 4ac < 0$.

Sum of roots $= -\frac{b}{a}$. Product of roots $= \frac{c}{a}$.

Graph of $y = ax^2 + bx + c$ is a parabola.

Standard Equations of Conic Sections.

Circle: $x^2 + y^2 = r^2$. Parabola: $y^2 = 4ax$; $x^2 = 4ay$.

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$.

Rectangular Hyperbola: $xy = \pm k^2$.

Ratio, Proportion, Variation.

If, $a : b = c : d$,

then,

(1) $a + b : b = c + d : d$;

(2) $a - b : b = c - d : d$;

(3) $a + b : a - b = c + d : c - d$;

(4) $a^n : b^n = c^n : d^n$.

If $a_1 : b_1 = a_2 : b_2 = a_3 : b_3 = \cdots,$

then any of these ratios $= \frac{pa_1 + qa_2 + ra_3 + \cdots}{pb_1 + qb_2 + rb_3 + \cdots}$,

where p, q, r are any multipliers;

also any of these ratios $= \sqrt[n]{\frac{a_1^n + a_2^n + a_3^n + \cdots}{b_1^n + b_2^n + b_3^n + \cdots}}$.

If $y \propto x$ then $y = kx$;

If $y \propto \frac{1}{x}$ then $y = \frac{k}{x}$, or $xy = k$.

Arithmetic Progression. (180.)

a = first term; d = common diff.; n = number of terms;
l = last or nth term; S = sum of n terms.

$$n\text{th term} = l = a + (n-1)d.$$

$$S = \frac{n}{2}(a + l) = n\left(a + \frac{n-1}{2}d\right).$$

$$\text{Arithmetic mean of } a \text{ and } b = \frac{a+b}{2}.$$

Geometric Progression. (184.)

$r =$ the ratio; a, n, l, S, as above.

$$n\text{th term} = l = ar^{n-1}.$$

$$S = a\frac{1-r^n}{1-r} = \frac{a-rl}{1-r}.$$

$$\text{Geometric mean of } a \text{ and } b = \sqrt{ab}.$$

$$\text{Sum of infinite geom. progr.} = \frac{a}{1-r}, \text{ if } |r| < 1.$$

Infinite Series. — Tests for convergence or divergence.

Series, $u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n + \cdots$.

Converges when the terms are alternately + and −, and steadily decrease toward zero (**199**).

Converges when the ratio $\frac{u_n}{u_{n-1}}$ becomes and remains numerically less than 1 for all values of n, provided always that $\lim u_n = 0$. (**202.**)

Diverges when the ratio $\frac{u_n}{u_{n-1}}$ becomes and remains greater than 1, or approaches 1 from the upper side. (**202.**)

Converges when its terms are numerically less than the corresponding terms of a series known to converge absolutely. (**201.**)

Diverges when its terms are all of like sign and are numerically greater than the corresponding terms of a known divergent series.

Test Series.

$$1 + x + x^2 + x^3 + \cdots \quad \begin{cases} \text{conv. when } |x| < 1; \\ \text{div. when } |x| \geqq 1. \end{cases}$$

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \quad \begin{cases} \text{conv. when } p > 1; \\ \text{div. when } p \leqq 1. \end{cases}$$

Derivatives. (**210.**)

$$D_x y \equiv \frac{dy}{dx} \equiv \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} \begin{cases} = \text{slope of tangent to curve } y = f(x). \\ = \text{rate of change of } y \text{ relative to } x. \end{cases}$$

Formulas for Differentiation. (211–2.)

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}. \qquad \frac{dc}{dx} = 0. \qquad \frac{d(cy)}{dx} = c\frac{dy}{dx}.$$

$$\frac{d(u+v+w+\cdots)}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \cdots.$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \qquad \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

when y is a function of u, and u a function of x.

$$\frac{dx^n}{dx} = nx^{n-1}. \qquad \frac{d\log x}{dx} = \frac{1}{x}. \qquad \frac{da^x}{dx} = a^x \log a.$$

$$\frac{d\sin x}{dx} = \cos x. \qquad \frac{d\cos x}{dx} = -\sin x.$$

$$\frac{d\tan x}{dx} = \sec^2 x. \qquad \frac{d\cot x}{dx} = -\csc^2 x.$$

$$\frac{d\sec x}{dx} = \sec x \tan x\, dx. \qquad \frac{d\csc x}{dx} = -\csc x \cot x.$$

$$\frac{d\sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}}. \qquad \frac{d\cos^{-1} x}{dx} = \frac{-1}{\sqrt{1-x^2}}.$$

$$\frac{d\tan^{-1} x}{dx} = \frac{1}{1+x^2}. \qquad \frac{d\cot^{-1} x}{dx} = \frac{-1}{1+x^2}.$$

$$\frac{d\sec^{-1} x}{dx} = \frac{1}{x\sqrt{x^2-1}}. \qquad \frac{d\csc^{-1} x}{dx} = \frac{-1}{x\sqrt{x^2-1}}.$$

Maclaurin's Series. (218.)

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \cdots.$$

Some Standard Series.

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots. \qquad \text{Always convergent.}$$

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots. \qquad \text{Always convergent.}$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots . \qquad \text{Always convergent.}$$

$$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots . \qquad \text{Convergent only if } -1 < x \leqq 1.$$

Theorem of Undetermined Coefficients. (233–4.)

If, for all values of x from $x = 0$ to $x = h$ where h is any number other than zero, we have

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = 0,$$

then $$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0 \cdots a_n = 0, \cdots .$$

If, for values of x as above, we have

$$a_0 + a_1x + a_2x^2 + \ldots = b_0 + b_1x + b_2x^2 + \ldots ,$$

then $$a_0 = b_0, \quad a_1 = b_1, \quad a_2 = b_2, \text{ etc.}$$

Partial Fractions. (235–8.) — The partial fractions may be determined according to the factors of the denominator of the given fraction by the following rules:

Form of factor:	Corresponding fraction or fractions:
$(ax+b)$,	$\frac{A}{ax+b}$.
$(ax+b)^n$,	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$.
(ax^2+bx+c),	$\frac{A_x+B}{ax^2+bx+c}$.
$(ax^2+bx+c)^m$,	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$.

Determinants. (240–9.)

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - b_1B_1 + c_1C_1$$

$$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

Here A_1, B_1, C_1, are the minors of a_1, b_1, c_1, respectively.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 A_1 - b_1 B_1 + c_1 C_1 - d_1 D_1,$$

where A_1, B_1, C_1, D_1 are the minors of a_1, b_1, c_1, d_1, respectively. Similarly for a determinant of any order.

Differences and Interpolation. (227–32.)

Let $u_0, u_1, u_2, \cdots$ be a given sequence, and let $\Delta_1 u_0, \Delta_2 u_0, \Delta_3 u_0, \cdots$ be the first terms of the successive difference columns. Also let ${}_nC_1, {}_nC_2, {}_nC_3, \cdots$ be the binomial coefficients, i.e.,

$${}_nC_1 = n, \quad {}_nC_2 = \frac{n(n-1)}{\underline{|2}}, \quad {}_nC_3 = \frac{n(n-1)(n-2)}{\underline{|3}}, \quad \text{etc.}$$

Let u_n be the nth term of the sequence and s_n the sum of its first n terms. Then

$$u_n = u_0 + {}_nC_1\Delta_1 u_0 + {}_nC_2\Delta_2 u_0 + {}_nC_3\Delta_3 u_0 + \cdots;$$
$$S_n = {}_nC_1 u_0 + {}_nC_2\Delta_1 u_0 + {}_nC_3\Delta_2 u_0 + {}_nC_4\Delta_4 u_0 + \cdots.$$

If $u_0 = f(x_0)$, $u_1 = f(x_0 + h)$, $u_2 = f(x_0 + 2h)$, $u_3 = f(x_0 + 3h)$, . . . , then

$$f(x_0 + nh) = f(x_0) + {}_nC_1\Delta_1 f(x_0) + {}_nC_2\Delta_2 f(x_0) + {}_nC_3\Delta_3 f(x_0) + \cdots.$$

Here n need not be an integer.

Useful Approximations. (224.)

When $x, y, u, v, \ldots$ are small (near 0) we have, approximately,

$$(1+x)(1+y) = 1 + x + y. \qquad \frac{1}{1+x} = 1 - x.$$
$$(1+x)(1-y) = 1 + x - y.$$
$$(1-x)(1-y) = 1 - x - y. \qquad \frac{1}{1-x} = 1 + x.$$
$$\frac{1+x}{1+y} = 1 + x - y. \quad \frac{(1+x)(1+y)\ldots}{(1+u)(1+v)\ldots} = 1 + x + y + \cdots - u - v - \cdots.$$

$$(1+x)^n = 1 + nx. \qquad \text{As special cases of this:}$$
$$\sqrt{1+x} = 1 + \tfrac{1}{2}x. \qquad \sqrt{1-x} = 1 - \tfrac{1}{2}x.$$
$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x. \qquad \frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x.$$
$$(1+x)^2 = 1 + 2x. \qquad (1-x)^2 = 1 - 2x.$$
$$e^x = 1 + x. \quad \log_e(1+x) = x. \quad \log_{10}(1+x) = .43x.$$
$$\sin x = \tan x = x \text{ (radians)}. \qquad \cos x = 1.$$

More accurately,

$$\sin x = x - \frac{x^3}{6}. \quad \cos x = 1 - \frac{x^2}{2}. \quad \tan x = x + \frac{x^3}{3}.$$

De Moivre's Theorem. (256.)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$
$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

The *n*th Roots of Unity. (259.)

$$x_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}; \qquad k = 0, 1, 2, \ldots, n-1.$$

Expansions of cos *n*θ and sin *n*θ. (260.)

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{\underline{|2}}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\cos^{n-4}\theta\sin^4\theta - \cdots.$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\theta\sin^3\theta + \cdots.$$

Exponential Values of sin *x* and cos *x*. (261.)

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}. \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

Hyperbolic Functions. (262.)

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \cdots.$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \cdots.$$

$$\tanh x = \frac{\sinh x}{\cosh x}. \qquad \coth x = \frac{\cosh x}{\sinh x}.$$

$$\text{sech}\, x = \frac{1}{\cosh x}. \qquad \text{csch}\, x = \frac{1}{\sinh x}.$$

Permutations and Combinations. (263–4.)

$${}_nP_r = n(n-1)(n-2)\ldots(n-r+1). \quad {}_nP_n = \underline{|n}.$$

$${}_nC_r = \frac{{}_nP_r}{\underline{|r}} = \frac{n(n-1)\ldots(n-r+1)}{\underline{|r}} = {}_nC_{n-r}.$$

Definitions. (**124, 132.**) — In right triangle ABC, whose sides are a, b, c [figure of (**124**)],

$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b},$$

$$\csc A = \frac{c}{a}, \quad \sec A = \frac{c}{b}, \quad \cot A = \frac{b}{a}.$$

$$\operatorname{vers} A = 1 - \cos A. \quad \operatorname{covers} A = 1 - \sin A.$$

More generally, if x be an angle of any magnitude, as XOP in the figure of (**132**),

$$\sin x = \frac{\text{ordinate}}{\text{distance}}, \quad \cos x = \frac{\text{abscissa}}{\text{distance}}, \quad \tan x = \frac{\text{ordinate}}{\text{abscissa}},$$

$$\csc x = \frac{\text{distance}}{\text{ordinate}}, \quad \sec x = \frac{\text{distance}}{\text{abscissa}}, \quad \cot x = \frac{\text{abscissa}}{\text{ordinate}}.$$

Relations between the Functions of an Angle. Formulas, Group A. (**137.**)

1. $\sin x = \frac{1}{\csc x}$.
2. $\cos x = \frac{1}{\sec x}$.
3. $\tan x = \frac{1}{\cot x}$.
4. $\tan x = \frac{\sin x}{\cos x}$.
5. $\cot x = \frac{\cos x}{\sin x}$.
6. $\sin^2 x + \cos^2 x = 1$.
7. $1 + \tan^2 x = \sec^2 x$.
8. $1 + \cot^2 x = \csc^2 x$.

Rules for expressing any function of any angle in terms of a function of an acute angle. (**139.**)

Any function of any angle x is numerically equal to the $\left\{\begin{matrix}\text{same function}\\ \text{co-function}\end{matrix}\right.$ of x increased or diminished by any $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right.$ multiple of 90°.

The sign of the result must be determined according to the quadrant of x.

Functions of $+x$ and $-x$. (140.)

$f(+x) = f(-x)$, when $f =$ cosine or secant.

$f(+x) = -f(-x)$, when $f =$ sine, cosecant, tangent, cotangent.

Angles Corresponding to a Given Function. (146.)

Let θ denote the smallest positive angle having a given function equal to a given number a. Then all angles such that

I. $\begin{cases} \sin x = a \\ \csc x = a \end{cases}$ are $x = 2n\pi + \theta$ and $(2n+1)\pi - \theta$;

II. $\begin{cases} \cos x = a \\ \sec x = a \end{cases}$ are $x = 2n\pi \pm \theta$;

III. $\begin{cases} \tan x = a \\ \cot x = a \end{cases}$ are $x = n\pi + \theta$.

Formulas, Group B. (155.)

9. $\sin(x+y) = \sin x \cos y + \cos x \sin y.$
10. $\cos(x+y) = \cos x \cos y - \sin x \sin y.$
11. $\sin(x-y) = \sin x \cos y - \cos x \sin y.$
12. $\cos(x-y) = \cos x \cos y + \sin x \sin y.$
13. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}.$
14. $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}.$
15. $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}.$
16. $\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}.$

Formulas, Group C. (157.)

Double Angle.

14. $\sin 2x = 2 \sin x \cos x.$

15. $\cos 2x = \cos^2 x - \sin^2 x,$
$= 1 - 2\sin^2 x,$
$= 2\cos^2 x - 1.$

16. $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}.$

Half-Angle.

17. $\sin \frac{1}{2} x = \pm \sqrt{\dfrac{1 - \cos x}{2}}.$

18. $\cos \frac{1}{2} x = \pm \sqrt{\dfrac{1 + \cos x}{2}}.$

19. $\tan \frac{1}{2} x = \pm \sqrt{\dfrac{1 - \cos x}{1 + \cos x}},$
$= \dfrac{1 - \cos x}{\sin x},$
$= \dfrac{\sin x}{1 - \cos x}.$

Formulas, Group D. (158.)

20. $$\sin u + \sin v = 2 \sin \frac{u+v}{2} \cos \frac{u-v}{2}.$$

21. $$\sin u - \sin v = 2 \cos \frac{u+v}{2} \sin \frac{u-v}{2}.$$

22. $$\cos u + \cos v = 2 \cos \frac{u+v}{2} \cos \frac{u-v}{2}.$$

23. $$\cos u - \cos v = -2 \sin \frac{u+v}{2} \sin \frac{u-v}{2}.$$

Solution of Plane Triangles

Right Triangles. — By means of the definitions of the trigonometric functions write an equation involving the two given parts and a required part; solve this for the required part.

Oblique Plane Triangles. (169–172.)

Law of Sines: 1. $a : b : c = \sin A : \sin B : \sin C$ **(169)**

Law of Cosines: 2. $a^2 = b^2 + c^2 - 2\,bc \cos A.$ **(170)**

Law of Tangents: 3. $\dfrac{a-b}{a+b} = \dfrac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$ **(171)**

Half-Angles: (172.)

Let $s = \frac{1}{2}(a+b+c)$ and $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}.$

4. $\sin \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}.$

5. $\cos \frac{1}{2} A = \sqrt{\dfrac{s(s-a)}{bc}}.$

6. $\tan \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}.$

7. $\tan \frac{1}{2} A = \dfrac{r}{s-a}.$

Solution of Oblique Plane Triangles. (173–8.)

Case I. Given two angles and a side. **(174)**
Use law of sines.

Case II. Given two sides and the included angle. **(175)**
Use law of tangents, then law of sines.

Case III. Given two sides and an opposite angle. **(176)**
Use law of sines. Ambiguous case.

Case IV. Given the three sides. **(177)**
Use one of the formulas (4), (5), (6), or (7) above, preferably the last one.

$$\text{Area} = \tfrac{1}{2}\, ab \sin C = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}. \qquad \textbf{(178)}$$

Spherical Trigonometry

Spherical Right Triangle. (313–6.) — Let A, B, C be the angles, and a, b, c the sides. Arrange the five parts a, b, co-B, co-c, co-A in circular order. These parts are then connected by Napier's Rules:

$$\text{sine of middle part} = \begin{cases} \text{product of cosines of opposite parts;} \\ \text{product of tangents of adjacent parts.} \end{cases}$$

To solve a spherical right triangle use Napier's Rules to write a formula involving the two given parts and a required part.

To solve a quadrantal triangle, solve its polar right triangle.

Spherical Oblique Triangles. (317–22.)

Law of Sines: $\sin a : \sin b : \sin c = \sin A : \sin B : \sin C$.

Law of Cosines: $\cos a = \cos b \cos c + \sin b \sin c \cos A$.

Half-Angles.

$$s = \frac{1}{2}(a+b+c); \ \tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

4. $$\sin\frac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}.$$

5. $$\cos\frac{1}{2}A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.$$

6. $$\tan\frac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}.$$

8. $$\tan\frac{1}{2}A = \frac{\tan r}{\sin(s-a)}.$$

Half-Sides.

$$S = \frac{1}{2}(A+B+C);\ \tan R = \sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}.$$

13. $$\sin\frac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}}.$$

14. $$\cos\frac{1}{2}a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}.$$

15. $$\tan\frac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}.$$

16. $$\tan\frac{1}{2}a = \tan R\cos(S-A).$$

Napier's Analogies.

19 $$\tan\frac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{1}{2}c.$$

20. $$\tan\frac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{1}{2}c.$$

21. $$\tan\frac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{1}{2}C.$$

22. $$\tan\frac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{1}{2}C.$$

Spherical Excess.

$$E = (A+B+C) - 180°.$$

23. $$\tan\frac{1}{2}E = \frac{\tan\frac{1}{2}a\tan\frac{1}{2}b\sin C}{1+\tan\frac{1}{2}a\tan\frac{1}{2}b\cos C}.$$

24. $$\tan\frac{1}{4}E = \sqrt{\tan\frac{1}{2}s\tan\frac{1}{2}(s-a)\tan\frac{1}{2}(s-b)\tan\frac{1}{2}(s-c)}.$$

$$\text{Area} = \frac{E\text{ (degrees)}}{720}\times 4\pi R^2 = E\text{ (radians)}\times R^2.$$

Solution of Spherical Oblique Triangle. (323.)

I. Given two sides and an opposite angle.
Use law of sines, then Napier's Analogies. Two solutions possible.

II. Given two angles and an opposite side.
As in I.

III. Given the three sides.
Use formulas for the half-angles.

IV. Given the three angles.
Use formulas for the half-sides.

V. Given two sides and their included angle.
Use Napier's Analogies, then law of sines.

VI. Given two angles and their included side.
As in V.

APPENDIX B

Explanation of the Tables and Their Use

TABLE I

This table gives the decimal part, or *mantissa*, of the logarithm of every positive number containing not more than three significant figures. The mantissas of the logarithms of numbers containing more than three significant figures are to be obtained by *interpolation* (**35**). The integral part, or *characteristic*, of the logarithm must be supplied by the computer, according to the position of the decimal point in the number.

Rules for Characteristics.

(**a**) When the number has n significant figures to the left of the decimal point, the characteristic of its logarithm is $n - 1$.

(**b**) When the number is a decimal with n ciphers between the decimal point and the first digit which is not zero, the characteristic of its logarithm is $9 - n$, and -10 must be supplied to complete the logarithm.

The reason for these rules will become evident when we consider an example.

Example. Let us find log 302. In the table find 30 in the left-hand column and run across the page horizontally to the column headed 2. There we find that

$$\text{mantissa of } \log 302 = .4800.$$

Now 302 lies between 100 and 1000, i.e. between 10^2 and 10^3. Hence, by the definition of a logarithm, log 302 must lie between 2 and 3. Therefore the characteristic is 2, and

$$\log 302 = 2.4800.$$

This is of course not the *exact* logarithm of 302, but only its value to four decimal places.

Writing the last equation in exponential form, we have

$$302 = 10^{2.4800}.$$

Multiplying both sides by 10,

$$3020 = 10 \times 10^{2.4800} = 10^{3.4800}. \quad \text{Hence, } \log 3020 = 3.4800.$$

Multiplying again by 10,

$$30200 = 10 \times 10^{3.4800} = 10^{4.4800}. \quad \text{Hence, } \log 30200 = 4.4800.$$

Therefore, where a number is multiplied by 10, the characteristic of its logarithm is increased by 1; the mantissa remains unchanged.

Dividing the above equation successively by 10, we obtain

$$\begin{aligned} 30.2 &= 10^{2.4800} \div 10 = 10^{1.4800}, \\ 3.02 &= 10^{1.4800} \div 10 = 10^{0.4800}, \\ .302 &= 10^{0.4800} \div 10 = 10^{0.4800-1}, \\ .0302 &= 10^{0.4800-1} \div 10 = 10^{0.4800-2}, \\ .00302 &= 10^{0.4800-2} \div 10 = 10^{0.4800-3}, \end{aligned}$$

and so on. As logarithmic equations these are:

$$\begin{aligned} \log 30.2 &= 1.4800, \\ \log 3.02 &= 0.4800, \\ \log .302 &= 0.4800 - 1 = 9.4800 - 10, \\ \log .0302 &= 0.4800 - 2 = 8.4800 - 10, \\ \log .00302 &= 0.4800 - 3 = 7.4800 - 10, \end{aligned}$$

and so on. The second form in the last three equations is used for convenience in computations; it is in accordance with rule (**b**).

To discuss rules (**a**) and (**b**) more generally, let m be any number. Then by the definition of a logarithm, when

	m lies between	$\log m$ lies between
(1)	1 and 10,	0 and 1,
(2)	10 and 100,	1 and 2,
(3)	100 and 1000,	2 and 3,
(4)	1000 and 10000,	3 and 4,

and so on. Therefore, when m has

(1) 1 digit to the left of the point, $\log m = 0.+ \cdots$;
(2) 2 digits to the left of the point, $\log m = 1.+ \cdots$;
(3) 3 digits to the left of the point, $\log m = 2.+ \cdots$;
(4) 4 digits to the left of the point, $\log m = 3.+ \cdots$;

and so on. Hence rule (**a**).

In the case of decimal numbers,

	when m lies between	log m lies between
(1)	1 and 0.1,	0 and − 1,
(2)	0.1 and 0.01,	− 1 and − 2,
(3)	0.01 and 0.001,	− 2 and − 3,
(4)	0.001 and 0.0001,	− 3 and − 4,

and so on. That is, when m is a decimal number in which

(1) no cipher follows the point, $\log m = 9.+ \cdots - 10$;
(2) 1 cipher follows the point, $\log m = 8.+ \cdots - 10$;
(3) 2 ciphers follow the point, $\log m = 7.+ \cdots - 10$;
(4) 3 ciphers follow the point, $\log m = 6.+ \cdots - 10$;

and so on. Hence rule **(b)**.

Interpolation. — *Example.* Find log 3024.
From the table,

mantissa of log 302 = .4800;
mantissa of log 303 = .4814; difference = .0014.

Assuming that the increase in the logarithm is proportional to the increase in the number, we have

$$\text{mantissa of } \log 3024 = .4800 + .4 \times .0014 = .4806.$$

The result is here given to the nearest unit in the fourth decimal place, $.4 \times .0014$ being taken equal to .0006 in place of .00056.

Proportional Parts. — For convenience in interpolation, the tabular differences greater than 20 are subdivided into tenths and tabulated under the heading "Prop. Parts." When the difference is less than 20, the interpolation is best made mentally. If it is desired, the table of proportional parts may be used when $d < 20$ by taking half the proportional part corresponding to double the difference.

Examples.

1. log 164.3 = ?

Mantissa of log 164 = .2148; $d = 27$,
Correction for .3 = 8
log 164.3 = 2.2156

2. $\log (164.3)^{\frac{2}{3}} = ?$

$$\log (164.3)^{\frac{2}{3}} = \tfrac{2}{3} \log 164.3.$$
$$= \tfrac{2}{3}(2.2156) = 1.4771.$$

3. $\log .01047 = ?$

$$\begin{aligned} \text{Mantissa of } \log 104 &= .0170; \qquad d = 42, \\ \text{Correction for } .7 &= \underline{\quad 29} \\ \log .01047 &= 8.0199 - 10 \end{aligned}$$

4. $\log \sqrt[3]{(.01047)^4} = ?$

$$\begin{aligned} \sqrt[3]{.01047^4} &= (.01047)^{\frac{4}{3}}, \\ \log \sqrt[3]{(.01047)^4} &= \tfrac{4}{3} \log (.01047), \\ &= \tfrac{4}{3} (8.0199 - 10). \\ 4\,(8.0199 - 10) &= 32.0796 - 40 = 22.0796 - 30. \\ \tfrac{1}{3}\,(22.0796 - 30) &= 7.3599 - 10. \end{aligned}$$

Note. When a logarithm which is followed by -10 is to be divided by a number, add and subtract a multiple of ten so that the quotient will come out in a form followed by -10. Thus:

$$\tfrac{1}{4}\,(8.2448 - 10) = \tfrac{1}{4}\,(38.2448 - 40) = 9.5612 - 10.$$

Anti-logarithm. — The number whose logarithm is x is called the *anti-logarithm* of x.

Thus, if $x = \log m$, then $m = \text{anti-log } x$.

Given a logarithm, to obtain the corresponding number (anti-logarithm).

Examples.

1. $\log m = 0.4806. \quad m = ?$

The given logarithm lies between the tabular logarithms .4800 and .4814, to which correspond the numbers 302 and 303 respectively. Thus we have

Number.	Mantissa of log.		
302	.4800	6	14
m	.4806		
303	.4814		

Hence, without regard to the decimal point, $m = 302 + \frac{6}{14} = 3024+$. Pointing off properly,

$$m = \text{anti-log } 0.4806 = 3.024+.$$

2. $\log m = 7.0959 - 10. \quad m = ?$

mantissa of log 124	= .0934	25	35
mantissa of log m	= .0959		
mantissa of log 125	= .0969		

Hence m has the sequence of figures

$$124 + \tfrac{25}{35} = 1247 +.$$

Pointing off properly,

$$m = \text{anti-log } (7.0959 - 10) = .001247+.$$

Note. The value of the quotient $\frac{25}{35}$ may be obtained from the column of Prop. Parts by finding the number of tenths of 35 required to equal 25. We have from this column,

$$.7 \times 35 = 24.5 \text{ and } .8 \times 35 = 28.0.$$

Hence we see that to make 25 we need a little more than $.7 \times 35$. A close approximation would be .71+, making $m = .0012471+$.

When the tabular difference is large, it is possible to obtain correctly more than four significant figures of a number when its four-place logarithm is given.

Cologarithm. — The *cologarithm* of a number is the logarithm of the reciprocal of the number.

Thus: $\text{colog}\, m = \log \frac{1}{m} = \log 1 - \log m = -\log m.$

In practice we usually write it in the form

$$\text{colog}\, m = -\log m = (10 - \log m) - 10.$$

Rule. To form the cologarithm of a number, subtract its logarithm from 10 and write -10 after the result.

Examples.

1. $\text{colog}\, 302 = (10 - \log 302) - 10$
$= (10 - 2.4800) - 10 = 7.5200 - 10.$

2. $\text{colog}\, .003024 = (10 - \log .003024) - 10$
$= (10 - [7.4806 - 10]) - 10 = 2.5194.$

Use of the Cologarithm.

Example. Calculate the value of $\frac{302 \times .415}{541 \times .0828}$.

Let m be the value of the given fraction. Then without the use of cologarithms the calculation is as follows.

$$\log m = \log 302 + \log .415 - \log 541 - \log .0828.$$

log 302 =	2.4800	log 541 =	2.7332
log .415 =	9.6180 − 10	log .0828 =	8.9180 − 10
	12.0980 − 10		11.6512 − 10
	11.6512 − 10		
log m =	0.4468,	m =	2.7975.

To use cologarithms, we write

$$m = 302 \times .415 \times \frac{1}{541} \times \frac{1}{.0828}.$$

$$\log m = \log 302 + \log .415 + \text{colog}\, 541 + \text{colog}\, .0828$$

log 302 =	2.4800
log .415 =	9.6180 − 10
colog 541 =	7.2668 − 10
colog .0828 =	1.0820
log m =	20.4468 − 20
m =	2.7975.

As a last example, we calculate the value of the quantity,

$$m = \sqrt{\frac{(.00812)^{\frac{2}{3}} \times (-471.2)^3}{(-522.3)^3 \times (.01242)^{\frac{3}{4}}}}.$$

To take account of the signs, which must be done independently of the logarithmic calculation, we note that the cube of a negative quantity occurs on both sides of the fraction; hence the sign of the fraction is plus.

We now write

$$\log m = \tfrac{1}{2}\left[\log (.00812)^{\frac{2}{3}} + \log (471.2)^3 + \text{colog}\,(522.3)^3 + \text{colog}\,(.01242)^{\frac{3}{4}}\right].$$

$\log .00812 = 7.9096 - 10$	$\log (.00812)^{\frac{2}{3}} = 8.6064 - 10$
$\log 471.2 = 2.6732$	$\log (471.2)^3 = 8.0196$
$\log 522.3 = 2.7179$	$\log (522.3)^3 = 8.1537$
$\log .01242 = 8.0941 - 10$	$\log (.01242)^{\frac{3}{4}} = 8.5706 - 10$

Hence

$$\begin{aligned} \log (.00812)^{\frac{2}{3}} &= 8.6064 - 10 \\ \log (471.2)^3 &= 8.0196 \\ \text{colog}\,(522.3)^3 &= 1.8463 - 10 \\ \text{colog}\,(.01242)^{\frac{3}{4}} &= 1.4294 \\ 2\,| & \underline{19.9017 - 20} \\ \log m &= 9.9508 - 10 \\ m &= .8929. \end{aligned}$$

Exercises. Verify the following equations:

1. $\log 7 = 0.8451.$
2. $\log 253 = 2.4031.$
3. $\log 253.5 = 2.4040.$
4. $\log .0253 = 8.4031 - 10.$
5. $\log .002533 = 7.4036 - 10$
6. $\log 6544 = 3.8158.$
7. $\log 4.007 = 0.6028.$
8. $\log .9995 = 9.9998 - 10.$
9. $\log \sqrt{766} = 1.4421.$
10. $\log \frac{1}{766} = 7.1158 - 10.$
11. $\log (.0022)^3 = 2.0272 - 10.$
12. $\log \sqrt[3]{.0022} = 9.1141 - 10.$
13. $\log (.01401)^{\frac{4}{5}} = 8.5171 - 10.$
14. $\log (.0003684)^{\frac{7}{3}} = 7.9820 - 20.$
15. $\text{colog}\, 200 = 7.6990 - 10.$
16. $\text{colog}\, .7 = 0.1549.$
17. $\text{colog}\, .0448 = 1.3487.$
18. $\text{colog}\, \sqrt{5475} = 8.1308 - 10.$

19. colog $(.0003684)^{\frac{7}{2}} = 12.0180.$

20. antilog $1.2222 = 16.68.$

21. antilog $3.6675 = 4650.$

22. antilog $0.4000 = 2.5118.$

23. antilog $(8.3250 - 10) = .021135.$

24. antilog $(6.9525 - 10) = .0008964.$

25. $(.748)^3 = .4185.$

26. $\sqrt[3]{-.0822} = -.4348.$

27. $(-6.213)^{\frac{2}{5}} = 2.076.$

28. $\dfrac{(-.1412)^2}{\sqrt[3]{-(.00475)}} = -.11858.$

29. $\dfrac{1}{(72.32)^{\frac{2}{3}}} = .05761.$

TABLE II.

This table gives the logarithms of the sine, cosine, tangent and cotangent of angles from 0° to 90°, at intervals of 10′.

When the angle is taken from the left-hand column of the page, the name of the function must be sought at the top of the page; when the angle is taken from the right-hand column of the page, the name of the function must be sought at the foot of the page.

When the function is numerically less than 1, -10 must be written after its tabular logarithm. This is the case with the sines and cosines of all angles between 0° and 90°, with tangents of angles between 0° and 45°, and with cotangents between 45° and 90°.

For convenience in interpolation the differences of the tabular logarithms are given, and these differences are subdivided into tenths in the column of proportional parts. Hence this column contains the corrections to the tabular logarithms for each minute of angle from 1′ to 9′ inclusive. These corrections are to be added when the logarithm increases with the angle, and they are to be subtracted when the logarithm decreases as the angle increases.

When the logarithm of a function of an angle greater than 90° is required, change to the equivalent function of an angle less than 90° (**139**). Algebraic signs must be adjusted independently of the logarithmic calculation, as in the use of Table I.

Seconds of arc must be reduced to the equivalent fractions of a minute of arc.

To obtain log sec x, take from the table colog cos x; for log csc x use colog sin x.

Examples.

1. log sin 20° 13′ = ?

$$\log \sin 20°\ 10' = 9.5375; \qquad d = 34.$$
$$d \text{ for } 3' \text{ (Prop. Parts)} = \quad 10.2$$
$$\log \sin 20°\ 13' = 9.5385 - 10.$$

2. $\log \cos 20° 13' = ?$

$$\begin{aligned} \log \cos 20° 10' &= 9.9725; \quad d = 4. \\ d \text{ for } 3' = 4 \times .3 &= \underline{\quad 1.2} \\ \log \cos 20° 13' &= 9.9724 - 10. \end{aligned}$$

3. $\log \tan 29° 47' = ?$

$$\begin{aligned} \log \tan 29° 40' &= 9.7556; \quad d = 29. \\ d \text{ for } 7' \text{ (Prop. Parts)} &= \underline{\quad 20.3} \\ \log \tan 29° 47' &= 9.7576 - 10 \end{aligned}$$

The same result may also be obtained by starting with log tan 29° 50′, thus:

$$\begin{aligned} \log \tan 29° 50' &= 9.7585; \quad d = 29. \\ d \text{ for } 3' &= \underline{\quad 8.7} \\ \log \tan 29° 47' &= 9.7576 - 10. \end{aligned}$$

As a rule, in interpolating start from the nearest tabular number.

4. $\log \cot 29° 47' = ?$

$$\begin{aligned} \log \cot 29° 50' &= 0.2415; \quad d = 29. \\ d \text{ for } 3' &= \underline{\quad 8.7} \\ \log \cot 29° 47' &= 0.2424. \end{aligned}$$

5. $\log \sin 58° 44' = ?$

$$\begin{aligned} \log \sin 58° 40' &= 9.9315; \quad d = 8. \\ d \text{ for } 4' &= \underline{\quad 3.2} \\ \log \sin 58° 44' &= 9.9318 - 10. \end{aligned}$$

6. $\log \tan 67° 23'.5 = ?$

$$\begin{aligned} \log \tan 67° 20' &= 0.3792; \quad d = 36. \\ d \text{ for } 3'.5 = 10.8 + 1.8 &= \underline{\quad 12.6} \\ \log \tan 67° 23'.5 &= 0.3805. \end{aligned}$$

Here we obtain d for 3′.5 from d for 3′ + d for 0′.5. Note that d for 0.5 is simply one-tenth of d for 5′.

7. $\log \cos 105° 51'.6 = ?$

$$\cos 105° 51'.6 = -\sin 15° 51'.6.$$

Neglecting the algebraic sign we have

$$\begin{aligned} \log \sin 15° 50' &= 9.4359; \quad d = 44. \\ d \text{ for } 1'.6 &= \underline{\quad 7.0} \\ \log \sin 15° 51'.6 &= 9.4366 - 10 = \log \cos 105° 51'.6. \end{aligned}$$

8. $\log \tan 250° 34'.3 = ?$

$$\begin{aligned} \tan 250° 34'.3 &= \tan 70° 34'.3. \\ \log \tan 70° 30' &= 0.4509; \quad d = 40. \\ d \text{ for } 4'.3 &= \underline{\quad 17.2} \\ \log \tan 70° 34'.3 &= 0.4526 = \log \tan 250° 34'.3. \end{aligned}$$

Angles near 0° or near 90°.

When an angle, x, lies near 0°, sin x, tan x, and cot x vary too rapidly with x to permit of accurate interpolation of their logarithms from the table. The same is true of cos x, tan x, and cot x, when x lies near 90°. We will show how accurate values of these logarithms may be obtained.

Let $$S = \log \frac{\sin x}{x} \quad \text{and} \quad T = \log \frac{\tan x}{x},$$

x being expressed in minutes of arc.

Then $$\log \sin x = \log x' + S,$$

and $$\log \tan x = \log x' + T.$$

When x is small the quantities S and T vary quite slowly with x. The values of S and T are given in the last column of the first page of Table II, x ranging from 0° to 5°; -10 is to be added to the tabular numbers there given.

To get log sin x, reduce x to minutes of arc and take log x' from Table I; to this logarithm add S.

To get log tan x, add T to log x'.

To get log cot x, first get log tan x and form the cologarithm of the result.

For, $$\log \cot x = \text{colog} \tan x.$$

To obtain log cos x, log tan x or log cot x, when x lies between 85° and 90°, calculate the co-function of the complementary angle by the method given above.

To find the angle from log sin x, log tan x or log cot x, when x lies near 0°, we use the relations

$$\log x' = \log \sin x - S;$$
$$\log x' = \log \tan x - T;$$
$$\log x' = -\log \cot x - T.$$

The necessary values of S and T can be obtained after finding an approximate value of x from Table II.

To find x from log cos x, log tan x, or log cot x, when x lies near 90°, replace

$$\log \cos x \quad \text{by} \quad \log \sin (90° - x);$$
$$\log \tan x \quad \text{by} \quad \log \cot (90° - x);$$
$$\log \cot x \quad \text{by} \quad \log \tan (90° - x).$$

Then $90° - x$ can be obtained by the method given above for angles near $0°$. Hence x is determined.

Examples.

1. Find log sin x, log tan x and log cot x when $x = 1° 22' 12''$.

$x = 1° 22' 12'' = 82'.2.$ $\qquad$ $\log x' = \log 82.2 = 1.9149.$

$\log x = 1.9149$ $\qquad$ $\log x = 1.9149$

$S = 6.4637 - 10$ $\qquad$ $T = 6.4638 - 10$

$\log \sin x = 8.3786 - 10$ $\qquad$ $\log \tan x = 8.3787 - 10$

$\log \cot x = \text{colog} \tan x = 1.6213.$

2. Find log cos x, log tan x and log cot x when $x = 89° 5' 50''$.

Let $\qquad y = 90° - x = 54' 10'' = 54'.17.$

Then log cos x, log tan x, log cot x are equal respectively to log sin y, log cot y, log tan y, which may be found as in example 1.

3. $\qquad \log \sin x = 8.2142; \quad x = ?$

From Table II, $x = 50' + ;$ $\quad$ hence $S = 6.4637 - 10.$

$\log \sin x = 8.2142 - 10$

$S = 6.4637 - 10$

$\log x' = 1.7505;$ $\qquad$ $x = 56'.30 = 56' 18''.$

4. $\qquad \log \tan x = 8.0804 - 10; \quad x = ?$

From Table II, $x = 40' + ;$ $\quad$ hence $T = 6.4638$

$\log \tan x = 8.0804 - 10$

$T = 6.4638 - 10$

$\log x' = 1.6166;$ $\qquad$ $x = 41'.36 = 41' 21''.6.$

5. $\qquad \log \cot x = 8.6276 - 10; \quad x = ?$

Let $\qquad y = 90° - x.$

Then $\qquad \log \tan y = \log \cot x = 8.6276 - 10.$

From Table II, $y = 2° 20' + ;$ $\quad$ hence $T = 6.4640.$

$\log \tan y = 8.6276 - 10$

$T = 6.4640 - 10$

$\log y' = 2.1636; \; y = 145'.73 = 2° 25' 44''.$

Hence $\qquad x = 90° - y = 87° 34' 16''.$

Let the student obtain the results required in the last five examples by direct interpolation from Table II.

Exercises. Verify the following equations:

1. $\log \sin 20° 40' = 9.5477 - 10.$

2. $\log \cos 66° 30' = 9.6007 - 10.$

3. $\log \tan 29° 35' = 9.7541 - 10.$

4. $\log \cot 37° 25' = 0.1163.$

5. $\log \sec 55° 50' = 0.2506.$

6. $\log \csc 44° 50' = 0.1518.$

7. $\log \tan 63° 27' = 0.3013.$

8. $\log \sin 81° 29' = 9.9952.$

9. $\log \sin 81° 31' = 9.9952.$

10. $\log \cos 81° 29' = 9.1706 - 10.$

11. $\log \cos 81° 31' = 9.1689 - 10.$

12. $\log \cot 9° 6' = 0.7954.$

13. $\log \sin 152° 27' = 9.6651 - 10.$

14. $\log \sin 2° 10' 10'' = 8.5781 - 10.$

15. $\log \tan 1° 34' 20'' = 8.4385 - 10.$

16. $\log \cot 0° 10' 22'' = 2.5206.$

17. $\log \cos 89° 28' 44'' = 7.9588 - 10.$

18. $\log \tan 88° 46' 14'' = 1.6683.$

19. log sin x = 9.7926; x = 38° 20′.
20. log sin x = 9.3548; x = 13° 5′.
21. log sin x = 9.8867; x = 50° 23′.
22. log cos x = 9.6030; x = 66° 22′.
23. log tan x = 0.6278; x = 77° 44′.5.
24. log cot x = 0.0906; x = 39° 4′.
25. log cot x = 0.6648; x = 12° 12′.5.
26. log sec x = 0.1374; x = 43° 13′.
27. log csc x = 0.2890; x = 30° 56′.
28. log sec x = 0.6680; x = 77° 35′.8.
29. log sin x = 8.3698; x = 1° 20′ 34″.
30. log tan x = 8.7659; x = 3° 20′ 18″.
31. log cot x = 1.2952; x = 2° 54′ 3″.
32. log cos x = 8.5387; x = 88° 1′ 8″.
33. log cot x = 7.9485; x = 89° 29′ 28″.
34. log csc x = 2.3549; x = 0° 15′ 11″.
35. log sec x = 1.5102; x = 88° 13′ 48″.

TABLE III

This table gives the numerical values of the six trigonometric functions of angles from 0° to 90° at intervals of 10′. The functions of intermediate angles are to be obtained by interpolation.

By using the tables inversely, an angle may be found, usually to the nearest minute, when a function of the angle is known to four decimal places.

TABLE IV

This is a conversion table for changing from sexagesimal to radian measure, and conversely. The entries are given to five decimal places in radians, corresponding nearly to 2″ in sexagesimal measure.

Examples.

1. Express 200° 44′ 36″ in radian measure.

200° = 3 × 60° + 20°

3 × 60° = 3 × 1.04720 =	3.14160 radians.
20° =	0.34907
44′ =	0.01280
36″ =	0.00017
200° 44′ 36″	= 3.50364 radians.

2. Express 3.50364 radians in sexagesimal measure.

3.0	radians =	171° 53′ 14″
0.5	" =	28° 38′ 52″
0.003	" =	10′ 19″
0.0006	" =	2′ 4″
0.00004	" =	8″
3.50364 radians	=	200° 44′ 37″

TABLE V

This table contains the values of a number of mathematical constants, generally to fifteen places of decimals.

TABLE VI

This table gives the values of the natural or Naperian logarithm of x, and of the ascending and descending exponential functions e^x and e^{-x}, from $x = 0$ to $x = 5$ at intervals of 0.05. As a rule the tabular entries are given to three decimal places.

TABLE VII

This table gives the values of n^2, n^3, $\sqrt{n}$, and $\sqrt[3]{n}$, for values of n from 1 to 100.

The direct use of the table requires no explanation. As an example of its inverse use we find the approximate value of $\sqrt[3]{320}$. We have

$$(6.8)^3 = 314.432 \quad (n = 68),$$
$$(6.9)^3 = 328.509 \quad (n = 69).$$

Hence, interpolating linearly,

$$(6.840)^3 = 320 \text{ approx., or } \sqrt{320} = 6.840+.$$

TABLES

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	43	42
1	4.3	4.2
2	8.6	8.4
3	12.9	12.6
4	17.2	16.8
5	21.5	21.0
6	25.8	25.2
7	30.1	29.4
8	34.4	33.6
9	38.7	37.8

	41	40
1	4.1	4.0
2	8.2	8.0
3	12.3	12.0
4	16.4	16.0
5	20.5	20.0
6	24.6	24.0
7	28.7	28.0
8	32.8	32.0
9	36.9	36.0

	39	38
1	3.9	3.8
2	7.8	7.6
3	11.7	11.4
4	15.6	15.2
5	19.5	19.0
6	23.4	22.8
7	27 3	26.6
8	31.2	30.4
9	35.1	34.2

	37	36
1	3.7	3.6
2	7.4	7.2
3	11.1	10.8
4	14.8	14.4
5	18.5	18.0
6	22.2	21.6
7	25.9	25.2
8	29.6	28.8
9	33.3	32.4

	35	34
1	3.5	3.4
2	7.0	6.8
3	10.5	10.2
4	14.0	13.6
5	17.5	17.0
6	21.0	20.4
7	24.5	23.8
8	28.0	27.2
9	31.5	30.6

	33	32
1	3.3	3.2
2	6.6	6.4
3	9.9	9.6
6	13.2	12.8
5	16.5	16.0
6	19.8	19.2
7	23.1	22.4
8	26.4	25.6
9	29.7	28.8

Prop. Parts

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9594	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	31	30	29	28	27	26	25	24	23	22	21
1	3.1	3.0	2.9	2.8	2.7	2.6	2.5	2.4	2.3	2.2	2.1
2	6.2	6.0	5.8	5.6	5.4	5.2	5.0	4.8	4.6	4.4	4.2
3	9.3	9.0	8.7	8.4	8.1	7.8	7.5	7.2	6.9	6.6	6.3
4	12.4	12.0	11.6	11.2	10.8	10.4	10.0	9.6	9.2	8.8	8.4
5	15.5	15.0	14.5	14.0	13.5	13.0	12.5	12.0	11.5	11.0	10.5
6	18.6	18.0	17.4	16.8	16.2	15.6	15.0	14.4	13.8	13.2	12.6
7	21.7	21.0	20.3	19.6	18.9	18.2	17.5	16.8	16.1	15.4	14.7
8	24.8	24.0	23.2	22.4	21.6	20.8	20.0	19.2	18.4	17.6	16.8
9	27.9	27.0	26.1	25.2	24.3	23.4	22.5	21.6	20.7	19.8	18.9

Prop. Parts

x	log sin	d	log cos	d	log tan	d	log cot	
0° 0′	$-\infty$		10.0000	0	$-\infty$		∞	90° 0′
10′	7.4637	3011	.0000	0	7.4637	3011	2.5363	50′
20′	7.7648	1760	.0000	0	.7648	1761	.2352	40′
30′	.9408	1250	.0000	0	.9409	1249	.0591	30′
40′	8.0658	969	.0000	0	8.0658	969	1.9342	20′
50′	.1627	792	.0000	1	.1627	792	.8373	10′
1° 0′	8.2419	669	9.9999	0	8.2419	670	1.7581	89° 0′
10′	.3088	580	.9999	0	.3089	580	.6911	50′
20′	.3668	511	.9999	0	.3669	512	.6331	40′
30′	.4179	458	9.9999	1	.4181	457	.5819	30′
40′	.4637	413	.9998	0	.4638	415	.5362	20′
50′	.5050	378	.9998	1	.5053	378	.4947	10′
2° 0′	8.5428	348	9.9997	0	8.5431	348	1.4569	88° 0′
10′	.5776	321	.9997	1	.5779	322	.4221	50′
20′	.6097	300	.9996	0	.6101	300	.3899	40′
30′	.6397	280	.9996	1	.6401	281	.3599	30′
40′	.6677	263	.9995	0	.6682	263	.3318	20′
50′	.6940	248	.9995	1	.6945	249	.3055	10′
3° 0′	8.7188	235	9.9994	1	8.7194	235	1.2806	87° 0′
10′	.7423	222	.9993	0	.7429	223	.2571	50′
20′	.7645	212	.9993	1	.7652	213	.2348	40′
30′	.7857	202	.9992	1	.7865	202	.2135	30′
40′	.8059	192	.9991	1	.8067	194	.1933	20′
50′	.8251	185	.9990	1	.8261	185	.1739	10′
4° 0′	8.8436	177	9.9989	0	8.8446	178	1.1554	86° 0′
10′	.8613	170	.9989	1	.8624	171	.1376	50′
20′	.8783	163	.9988	1	.8795	165	.1205	40′
30′	.8946	158	.9987	1	.8960	158	.1040	30′
40′	.9104	152	.9986	1	.9118	154	.0882	20′
50′	.9256	147	.9985	2	.9272	148	.0728	10′
5° 0′	8.9403	142	9.9983	1	8.9420	143	1.0580	85° 0′
10′	.9545	137	.9982	1	.9563	138	.0437	50′
20′	.9682	134	.9981	1	.9701	135	.0299	40′
30′	.9816	129	.9980	1	.9836	130	.0164	30′
40′	.9945	125	.9979	2	.9966	127	.0034	20′
50′	9.0070	122	.9977	1	9.0093	123	0.9907	10′
6° 0′	9.0192	119	9.9976	1	9.0216	120	0.9784	84° 0′
10′	.0311	115	.9975	2	.0336	117	.9664	50′
20′	.0426	113	.9973	1	.0453	114	.9547	40′
30′	.0539	109	.9972	1	.0567	111	.9433	30′
40′	.0648	107	.9971	2	.0678	108	.9322	20′
50′	.0755	104	.9969	1	.0786	105	.9214	10′
7° 0′	9.0859	102	9.9968	2	9.0891	104	0.9109	83° 0′
10′	.0961	99	.9966	2	.0995	101	.9005	50′
20′	.1060	97	.9964	1	.1096	98	.8904	40′
30′	.1157		.9963		.1194		.8806	30′
	log cos	d	log sin	d	log cot	d	log tan	x

Small Angles

x	S	T
<1°	6.4637	6.4637
1°	6.4637	6.4638
2°	6.4636	6.4639
3°	6.4635	6.4641
4°	6.4634	6.4644
5°	6.4631	6.4649

Prop. Parts.

	113	111	109
1	11.3	11.1	10.9
2	22.6	22.2	21.8
3	33.9	33.3	32.7
4	45.2	44.4	43.6
5	56.5	55.5	54.5
6	67.8	66.6	65.4
7	79.1	77.7	76.3
8	90.4	88.8	87.2
9	101.7	99.9	98.1

	108	107	105
1	10.8	10.7	10.5
2	21.6	21.4	21.0
3	32.4	32.1	31.5
4	43.2	42.8	42.0
5	54.0	53.5	52.5
6	64.8	64.2	63.0
7	75.6	74.9	73.5
8	86.4	85.6	84.0
9	97.2	96.3	94.5

	104	102	101
1	10.4	10.2	10.1
2	20.8	20.4	20.2
3	31.2	30.6	30.3
4	41.6	40.8	40.4
5	52.0	51.0	50.5
6	62.4	61.2	60.6
7	72.8	71.4	70.7
8	83.2	81.6	80.8
9	93.6	91.8	90.9

	99	98	97	95
1	9.9	9.8	9.7	9.5
2	19.8	19.6	19.4	19.0
3	29.7	29.4	29.1	28.5
4	39.6	39.2	38.8	38.0
5	49.5	49.0	48.5	47.5
6	59.4	58.8	58.2	57.0
7	69.3	68.6	67.9	66.5
8	79.2	78.4	77.6	76.0
9	89.1	88.2	87.3	85.5

	143	142	138	137	135	134	130	129	127	125	123	122	119	117	115	114
1	14.3	14.2	13.8	13.7	13.5	13.4	13.0	12.9	12.7	12.5	12.3	12.2	11.9	11.7	11.5	11.4
2	28.6	28.4	27.6	27.4	27.0	26.8	26.0	25.8	25.4	25.0	24.6	24.4	23.8	23.4	23.0	22.8
3	42.9	42.6	41.4	41.1	40.5	40.2	39.0	38.7	38.1	37.5	36.9	36.6	35.7	35.1	34.5	34.2
4	57.2	56.8	55.2	54.8	54.0	53.6	52.0	51.6	50.8	50.0	49.2	48.8	47.6	46.8	46.0	45.6
5	71.5	71.0	69.0	68.5	67.5	67.0	65.0	64.5	63.5	62.5	61.5	61.0	59.5	58.5	57.5	57.0
6	85.8	85.2	82.8	82.2	81.0	80.4	78.0	77.4	76.2	75.0	73.8	73.2	71.4	70.2	69.0	68.4
7	100.1	99.4	96.6	95.9	94.5	93.8	91.0	90.3	88.9	87.5	86.1	85.4	83.3	81.9	80.5	79.8
8	114.4	113.6	110.4	109.6	108.0	107.2	104.0	103.2	101.6	100.0	98.4	97.6	95.2	93.6	92.0	91.2
9	128.7	127.8	124.2	123.3	121.5	120.6	117.0	116.1	114.3	112.5	110.7	109.8	107.1	105.3	103.5	102.6

x	log sin	d	log cos	d	log tan	d	log cot	
30′	9.1157	95	9.9963	2	9.1194	97	0.8806	30′
40′	.1252	93	.9961	2	.1291	94	.8709	20′
50′	.1345	91	.9959	1	.1385	93	.8615	10′
8° 0′	9.1436	89	9.9958	2	9.1478	91	0.8522	82° 0′
10′	.1525	87	.9956	2	.1569	89	.8431	50′
20′	.1612	85	.9954	2	.1658	87	.8342	40′
30′	.1697	84	.9952	2	.1745	86	.8255	30′
40′	.1781	82	.9950	2	.1831	84	.8169	20′
50′	.1863	80	.9948	2	.1915	82	.8085	10′
9° 0′	9.1943	79	9.9946	2	9.1997	81	0.8003	81° 0′
10′	.2022	78	.9944	2	.2078	80	.7922	50′
20′	.2100	76	.9942	2	.2158	78	.7842	40′
30′	.2176	75	.9940	2	.2236	77	.7764	30′
40′	.2251	73	.9938	2	.2313	76	.7687	20′
50′	.2324	73	.9936	2	.2389	74	.7611	10′
10° 0′	9.2397	71	9.9934	3	9.2463	73	0.7537	80° 0′
10′	.2468	70	.9931	2	.2536	73	.7464	50′
20′	.2538	68	.9929	2	.2609	71	.7391	40′
30′	.2606	68	.9927	3	.2680	70	.7320	30′
40′	.2674	66	.9924	2	.2750	69	.7250	20′
50′	.2740	66	.9922	3	.2819	68	.7181	10′
11° 0′	9.2806	64	9.9919	2	9.2887	66	0.7113	79° 0′
10′	.2870	64	.9917	3	.2953	67	.7047	50′
20′	.2934	63	.9914	2	.3020	65	.6980	40′
30′	.2997	61	.9912	3	.3085	64	.6915	30′
40′	.3058	61	.9909	2	.3149	63	.6851	20′
50′	.3119	60	.9907	3	.3212	63	.6788	10′
12° 0′	9.3179	59	9.9904	3	9.3275	61	0.6725	78° 0′
10′	.3238	58	.9901	2	.3336	61	.6664	50′
20′	.3296	57	.9899	3	.3397	61	.6603	40′
30′	.3353	57	.9896	3	.3458	59	.6542	30′
40′	.3410	56	.9893	3	.3517	59	.6483	20′
50′	.3466	55	.9890	3	.3576	58	.6424	10′
13° 0′	9.3521	54	9.9887	3	9.3634	57	0.6366	77° 0′
10′	.3575	54	.9884	3	.3691	57	.6309	50′
20′	.3629	53	.9881	3	.3748	56	.6252	40′
30′	.3682	52	.9878	3	.3804	55	.6196	30′
40′	.3734	52	.9875	3	.3859	55	.6141	20′
50′	.3786	51	.9872	3	.3914	54	.6086	10′
14° 0′	9.3837	50	9.9869	3	9.3968	53	0.6032	76° 0′
10′	.3887	50	.9866	3	.4021	53	.5979	50′
20′	.3937	49	.9863	4	.4074	53	.5926	40′
30′	.3986	49	.9859	3	.4127	51	.5873	30′
40′	.4035	48	.9856	3	.4178	52	.5822	20′
50′	.4083	47	.9853	4	.4230	51	.5770	10′
15° 0′	9.4130		9.9849		9.4281		0.5719	75° 0′
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	73	71	70	69	68
1	7.3	7.1	7.0	6.9	6.8
2	14.6	14.2	14.0	13.8	13.6
3	21.9	21.3	21.0	20.7	20.4
4	29.2	28.4	28.0	27.6	27.2
5	36.5	35.5	35.0	34.5	34.0
6	43.8	42.6	42.0	41.4	40.8
7	51.1	49.7	49.0	48.3	47.6
8	58.4	56.8	56.0	55.2	54.4
9	65.7	63.9	63.0	62.1	61.2

	67	66	65	64	63
1	6.7	6.6	6.5	6.4	6.3
2	13.4	13.2	13.0	12.8	12.6
3	20.1	19.8	19.5	19.2	18.9
4	26.8	26.4	26.0	25.6	25.2
5	33.5	33.0	32.5	32.0	31.5
6	40.2	39.6	39.0	38.4	37.8
7	46.9	46.2	45.5	44.8	44.1
8	53.6	52.8	52.0	51.2	50.4
9	60.3	59.4	58.5	57.6	56.7

	61	60	59	58	57
1	6.1	6.0	5.9	5.8	5.7
2	12.2	12.0	11.8	11.6	11.4
3	18.3	18.0	17.7	17.4	17.1
4	24.4	24.0	23.6	23.2	22.8
5	30.5	30.0	29.5	29.0	28.5
6	36.6	36.0	35.4	34.8	34.2
7	42.7	42.0	41.3	40.6	39.9
8	48.8	48.0	47.2	46.4	45.6
9	54.9	54.0	53.1	52.2	51.3

	56	55	54	53	52
1	5.6	5.5	5.4	5.3	5.2
2	11.2	11.0	10.8	10.6	10.4
3	16.8	16.5	16.2	15.9	15.6
4	22.4	22.0	21.6	21.2	20.8
5	28.0	27.5	27.0	26.5	26.0
6	33.6	33.0	32.4	31.8	31.2
7	39.2	38.5	37.8	37.1	36.4
8	44.8	44.0	43.2	42.4	41.6
9	50.4	49.5	48.6	47.7	46.8

	51	50	48	47
1	5.1	5.0	4.8	4.7
2	10.2	10.0	9.6	9.4
3	15.3	15.0	14.4	14.1
4	20.4	20.0	19.2	18.8
5	25.5	25.0	24.0	23.5
6	30.6	30.0	28.8	28.2
7	35.7	35.0	33.6	32.9
8	40.8	40.0	38.4	37.6
9	45.9	45.5	43.2	42.3

	97	94	93	91	89	87	86	85	84	82	81	79	78	77	76	75	74
1	9.7	9.4	9.3	9.1	8.9	8.7	8.6	8.5	8.4	8.2	8.1	7.9	7.8	7.7	7.6	7.5	7.4
2	19.4	18.8	18.6	18.2	17.8	17.4	17.2	17.0	16.8	16.4	16.2	15.8	15.6	15.4	15.2	15.0	14.8
3	29.1	28.2	27.9	27.3	26.7	26.1	25.8	25.5	25.2	24.6	24.3	23.7	23.4	23.1	22.8	22.5	22.2
4	38.8	37.6	37.2	36.4	35.6	34.8	34.4	34.0	33.6	32.8	32.4	31.6	31.2	30.8	30.4	30.0	29.6
5	48.5	47.0	46.5	45.5	44.5	43.5	43.0	42.5	42.0	41.0	40.5	39.5	39.0	38.5	38.0	37.5	37.0
6	58.2	56.4	55.8	54.6	53.4	52.2	51.6	51.0	50.4	49.2	48.6	47.4	46.8	46.2	45.6	45.0	44.4
7	67.9	65.8	65.1	63.7	62.3	60.9	60.2	59.5	58.8	57.4	56.7	55.3	54.6	53.9	53.2	52.5	51.8
8	77.6	75.2	74.4	72.8	71.2	69.6	68.8	68.0	67.2	65.6	64.8	63.2	62.4	61.6	60.8	60.0	59.2
9	87.3	84.6	83.7	81.9	80.1	78.3	77.4	76.5	75.6	73.8	72.9	71.1	70.2	69.3	68.4	67.5	66.6

Log.trig. funct's. 0° – 15° 90° – 75°
15° – 3 75° – 6
0° – 45° 0° – 45°
Nat. tr funct' 0° – 1 90° – 7
15° – 3 75° – 6
30° – 4 60° – 4
Radia to degree and co versel Math const'
Log$_e$ x, e^x, e^{-x}, n^2, n^3, $\sqrt{n}$, $\sqrt[3]{n}$

x	log sin	d	log cos	d	log tan	d	log cot	
15° 0′	9.4130	47	9.9849	3	9.4281	50	0.5719	75° 0′
10′	.4177	46	.9846	3	.4331	50	.5669	50′
20′	.4223	46	.9843	4	.4381	49	.5619	40′
30′	.4269	45	.9839	3	.4430	49	.5570	30′
40′	.4314	45	.9836	4	.4479	48	.5521	20′
50′	.4359	44	.9832	4	.4527	48	.5473	10′
16° 0′	9.4403	44	9.9828	3	9.4575	47	0.5425	74° 0′
10′	.4447	44	.9825	4	.4622	47	.5378	50′
20′	.4491	42	.9821	4	.4669	47	.5331	40′
30′	.4533	43	.9817	3	.4716	46	.5284	30′
40′	.4576	42	.9814	4	.4762	46	.5238	20′
50′	.4618	41	.9810	4	.4808	45	.5192	10′
17° 0′	9.4659	41	9.9806	4	9.4853	45	0.5147	73° 0′
10′	.4700	41	.9802	4	.4898	45	.5102	50′
20′	.4741	40	.9798	4	.4943	44	.5057	40′
30′	.4781	40	.9794	4	.4987	44	.5013	30′
40′	.4821	40	.9790	4	.5031	44	.4969	20′
50′	.4861	39	.9786	4	.5075	43	.4925	10′
18° 0′	9.4900	39	9.9782	4	9.5118	43	0.4882	72° 0′
10′	.4939	38	.9778	4	.5161	42	.4839	50′
20′	.4977	38	.9774	4	.5203	42	.4797	40′
30′	.5015	37	.9770	5	.5245	42	.4755	30′
40′	.5052	38	.9765	4	.5287	42	.4713	20′
50′	.5090	36	.9761	4	.5329	41	.4671	10′
19° 0′	9.5126	37	9.9757	5	9.5370	41	0.4630	71° 0′
10′	.5163	36	.9752	4	.5411	40	.4589	50′
20′	.5199	36	.9748	5	.5451	40	.4549	40′
30′	.5235	35	.9743	4	.5491	40	.4509	30′
40′	.5270	36	.9739	5	.5531	40	.4469	20′
50′	.5306	35	.9734	4	.5571	40	.4429	10′
20° 0′	9.5341	34	9.9730	5	9.5611	39	0.4389	70° 0′
10′	.5375	34	.9725	4	.5650	39	.4350	50′
20′	.5409	34	.9721	5	.5689	38	.4311	40′
30′	.5443	34	.9716	5	.5727	39	.4273	30′
40′	.5477	33	.9711	5	.5766	38	.4234	20′
50′	.5510	33	.9706	4	.5804	38	.4196	10′
21° 0′	9.5543	33	9.9702	5	9.5842	37	0.4158	69° 0′
10′	.5576	33	.9697	5	.5879	38	.4121	50′
20′	.5609	32	.9692	5	.5917	37	.4083	40′
30′	.5641	32	.9687	5	.5954	37	.4046	30′
40′	.5673	31	.9682	5	.5991	37	.4009	20′
50′	.5704	32	.9677	5	.6028	36	.3972	10′
22° 0′	9.5736	31	9.9672	5	9.6064	36	0.3936	68° 0′
10′	.5767	31	.9667	6	.6100	36	.3900	50′
20′	.5798	30	.9661	5	.6136	36	.3864	40′
30′	.5828		.9656		.6172		.3828	30′
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	50	49	48	47
1	5.0	4.9	4.8	4.7
2	10.0	9.8	9.6	9.4
3	15.0	14.7	14.4	14.1
4	20.0	19.6	19.2	18.8
5	25.0	24.5	24.0	23.5
6	30.0	29.4	28.8	28.2
7	35.0	34.3	33.6	32.9
8	40.0	39.2	38.4	37.6
9	45.0	44.1	43.2	42.3

	46	45	44	43
1	4.6	4.5	4.4	4.3
2	9.2	9.0	8.8	8.6
3	13.8	13.5	13.2	12.9
4	18.4	18.0	17.6	17.2
5	23.0	22.5	22.0	21.5
6	27.6	27.0	26.4	25.8
7	32.2	31.5	30.8	30.1
8	36.8	36.0	35.2	34.4
9	41.4	40.5	39.6	38.7

	42	41	40	39
1	4.2	4.1	4.0	3.9
2	8.4	8.2	8.0	7.8
3	12.6	12.3	12.0	11.7
4	16.8	16.4	16.0	15.6
5	21.0	20.5	20.0	19.5
6	25.2	24.6	24.0	23.4
7	29.4	28.7	28.0	27.3
8	33.6	32.8	32.0	31.2
9	37.8	36.9	36.0	35.1

	38	37	36	35
1	3.8	3.7	3.6	3.5
2	7.6	7.4	7.2	7.0
3	11.4	11.1	10.8	10.5
4	15.2	14.8	14.4	14.0
5	19.0	18.5	18.0	17.5
6	22.8	22.2	21.6	21.0
7	26.6	25.9	25.2	24.5
8	30.4	29.6	28.8	28.0
9	34.2	33.3	32.4	31.5

	34	33	32	31
1	3.4	3.3	3.2	3.1
2	6.8	6.6	6.4	6.2
3	10.2	9.9	9.6	9.3
4	13.6	13.2	12.8	12.4
5	17.0	16.5	16.0	15.5
6	20.4	19.8	19.2	18.6
7	23.8	23.1	22.4	21.7
8	27.2	26.4	25.6	24.8
9	30.6	29.7	28.8	27.9

Prop. Parts

x	log sin	d	log cos	d	log tan	d	log cot	
30′	9.5828	31	9.9656	5	9.6172	36	0.3828	30′
40′	.5859	30	.9651	5	.6208	35	.3792	20′
50′	.5889	30	.9646	6	.6243	36	.3757	10′
23° 0′	9.5919	29	9.9640	5	9.6279	35	0.3721	**67° 0′**
10′	.5948	30	.9635	6	.6314	34	.3686	50′
20′	.5978	29	.9629	5	.6348	35	.3652	40′
30′	.6007	29	.9624	6	.6383	34	.3617	30′
40′	.6036	29	.9618	5	.6417	35	.3583	20′
50′	.6065	28	.9613	6	.6452	34	.3548	10′
24° 0′	9.6093	28	9.9607	5	9.6486	34	0.3514	**66° 0′**
10′	.6121	28	.9602	6	.6520	33	.3480	50′
20′	.6149	28	.9596	6	.6553	34	.3447	40′
30′	.6177	28	.9590	6	.6587	33	.3413	30′
40′	.6205	27	.9584	5	.6620	34	.3380	20′
50′	.6232	27	.9579	6	.6654	33	.3346	10′
25° 0′	9.6259	27	9.9573	6	9.6687	33	0.3313	**65° 0′**
10′	.6286	27	.9567	6	.6720	32	.3280	50′
20′	.6313	27	.9561	6	.6752	33	.3248	40′
30′	.6340	26	.9555	6	.6785	32	.3215	30′
40′	.6366	26	.9549	6	.6817	33	.3183	20′
50′	.6392	26	.9543	6	.6850	32	.3150	10′
26° 0′	9.6418	26	9.9537	7	9.6882	32	0.3118	**64° 0′**
10′	.6444	26	.9530	6	.6914	32	.3086	50′
20′	.6470	25	.9524	6	.6946	31	.3054	40′
30′	.6495	26	.9518	6	.6977	32	.3023	30′
40′	.6521	25	.9512	7	.7009	31	.2991	20′
50′	.6546	24	.9505	6	.7040	32	.2960	10′
27° 0′	9.6570	25	9.9499	7	9.7072	31	0.2928	**63° 0′**
10′	.6595	25	.9492	6	.7103	31	.2897	50′
20′	.6620	24	.9486	7	.7134	31	.2866	40′
30′	.6644	24	.9479	6	.7165	31	.2835	30′
40′	.6668	24	.9473	7	.7196	30	.2804	20′
50′	.6692	24	.9466	7	.7226	31	.2774	10′
28° 0′	9.6716	24	9.9459	6	9.7257	30	0.2743	**62° 0′**
10′	.6740	23	.9453	7	.7287	30	.2713	50′
20′	.6763	24	.9446	7	.7317	31	.2683	40′
30′	.6787	23	.9439	7	.7348	30	.2652	30′
40′	.6810	23	.9432	7	.7378	30	.2622	20′
50′	.6833	23	.9425	7	.7408	30	.2592	10′
29° 0′	9.6856	22	9.9418	7	9.7438	29	0.2562	**61° 0′**
10′	.6878	23	.9411	7	.7467	30	.2533	50′
20′	.6901	22	.9404	7	.7497	29	.2503	40′
30′	.6923	23	.9397	7	.7526	30	.2474	30′
40′	.6946	22	.9390	7	.7556	29	.2444	20′
50′	.6968	22	.9383	8	.7585	29	.2415	10′
30° 0′	9.6990		9.9375		9.7614		0.2386	**60° 0′**
	log cos	d	log sin	d	log cot	d	log tan	*x*

Prop. Parts

	36	35	34	33	32	31	30	29	28	27	26	25	24	23	22
1	3.6	3.5	3.4	3.3	3.2	3.1	3.0	2.9	2.8	2.7	2.6	2.5	2.4	2.3	2.2
2	7.2	7.0	6.8	6.6	6.4	6.2	6.0	5.8	5.6	5.4	5.2	5.0	4.8	4.6	4.4
3	10.8	10.5	10.2	9.9	9.6	9.3	9.0	8.7	8.4	8.1	7.8	7.5	7.2	6.9	6.6
4	14.4	14.0	13.6	13.2	12.8	12.4	12.0	11.6	11.2	10.8	10.4	10.0	9.6	9.2	8.8
5	18.0	17.5	17.0	16.5	16.0	15.5	15.0	14.5	14.0	13.5	13.0	12.5	12.0	11.5	11.0
6	21.6	21.0	20.4	19.8	19.2	18.6	18.0	17.4	16.8	16.2	15.6	15.0	14.4	13.8	13.2
7	25.2	24.5	23.8	23.1	22.4	21.7	21.0	20.3	19.6	18.9	18.2	17.5	16.8	16.1	15.4
8	28.8	28.0	27.2	26.4	25.6	24.8	24.0	23.2	22.4	21.6	20.8	20.0	19.2	18.4	17.6
9	32.4	31.5	30.6	29.7	28.8	27.9	27.0	26.1	25.2	24.3	23.4	22.5	21.6	20.7	19.8

Prop. Parts

x	log sin	d	log cos	d	log tan	d	log cot	
30° **0′**	9.6990	22	9.9375	7	9.7614	30	0.2386	**60°** **0′**
10′	.7012	21	.9368	7	.7644	29	.2356	50′
20′	.7033	22	.9361	8	.7673	28	.2327	40′
30′	.7055	21	.9353	7	.7701	29	.2299	30′
40′	.7076	21	.9346	8	.7730	29	.2270	20′
50′	.7097	21	.9338	7	.7759	29	.2241	10′
31° **0′**	9.7118	21	9.9331	8	9.7788	28	0.2212	**59°** **0′**
10′	.7139	21	.9323	8	.7816	29	.2184	50′
20′	.7160	21	.9315	7	.7845	28	.2155	40′
30′	.7181	20	.9308	8	.7873	29	.2127	30′
40′	.7201	21	.9300	8	.7902	28	.2098	20′
50′	.7222	20	.9292	8	.7930	28	.2070	10′
32° **0′**	9.7242	20	9.9284	8	9.7958	28	0.2042	**58°** **0′**
10′	.7262	20	.9276	8	.7986	28	.2014	50′
20′	.7282	20	.9268	8	.8014	28	.1986	40′
30′	.7302	20	.9260	8	.8042	28	.1958	30′
40′	.7322	20	.9252	8	.8070	27	.1930	20′
50′	.7342	19	.9244	8	.8097	28	.1903	10′
33° **0′**	9.7361	19	9.9236	8	9.8125	28	0.1875	**57°** **0′**
10′	.7380	20	.9228	9	.8153	27	.1847	50′
20′	.7400	19	.9219	8	.8180	28	.1820	40′
30′	.7419	19	.9211	8	.8208	27	.1792	30′
40′	.7438	19	.9203	9	.8235	28	.1765	20′
50′	.7457	19	.9194	8	.8263	27	.1737	10′
34° **0′**	9.7476	18	9.9186	9	9.8290	27	0.1710	**56°** **0′**
10′	.7494	19	.9177	8	.8317	27	.1683	50′
20′	.7513	18	.9169	9	.8344	27	.1656	40′
30′	.7531	19	.9160	9	.8371	27	.1629	30′
40′	.7550	18	.9151	9	.8398	27	.1602	20′
50′	.7568	18	.9142	8	.8425	27	.1575	10′
35° **0′**	9.7586	18	9.9134	9	9.8452	27	0.1548	**55°** **0′**
10′	.7604	18	.9125	9	.8479	27	.1521	50′
20′	.7622	18	.9116	9	.8506	27	.1494	40′
30′	.7640	17	.9107	9	.8533	26	.1467	30′
40′	.7657	18	.9098	9	.8559	27	.1441	20′
50′	.7675	17	.9089	9	.8586	27	.1414	10′
36° **0′**	9.7692	18	9.9080	10	9.8613	26	0.1387	**54°** **0′**
10′	.7710	17	.9070	9	.8639	27	.1361	50′
20′	.7727	17	.9061	9	.8666	26	.1334	40′
30′	.7744	17	.9052	10	.8692	26	.1308	30′
40′	.7761	17	.9042	9	.8718	27	.1282	20′
50′	.7778	17	.9033	10	.8745	26	.1255	10′
37° **0′**	9.7795	16	9.9023	9	9.8771	26	0.1229	**53°** **0′**
10′	.7811	17	.9014	10	.8797	27	.1203	50′
20′	.7828	16	.9004	9	.8824	26	.1176	40′
30′	.7844		.8995		.8850		.1150	30′
	log cos	d	log sin	d	log cot	d	log tan	*x*

Prop. Parts

	30	**29**	**28**
1	3.0	2.9	2.8
2	6.0	5.8	5.6
3	9.0	8.7	8.4
4	12.0	11.6	11.2
5	15.0	14.5	14.0
6	18.0	17.4	16.8
7	21.0	20.3	19.6
8	24.0	23.2	22.4
9	27.0	26.1	25.2

	27	**26**	**22**
1	2.7	2.6	2.2
2	5.4	5.2	4.4
3	8.1	7.8	6.6
4	10.8	10.4	8.8
5	13.5	13.0	11.0
6	16.2	15.6	13.2
7	18.9	18.2	15.4
8	21.6	20.8	17.6
9	24.3	23.4	19.8

	21	**20**	**19**
1	2.1	2.0	1.9
2	4.2	4.0	3.8
3	6.3	6.0	5.7
4	8.4	8.0	7.6
5	10.5	10.0	9.5
6	12.6	12.0	11.4
7	14.7	14.0	13.3
8	16.8	16.0	15.2
9	18.9	18.0	17.1

	18	**17**	**16**
1	1.8	1.7	1.6
2	3.6	3.4	3.2
3	5.4	5.1	4.8
4	7.2	6.8	6.4
5	9.0	8.5	8.0
6	10.8	10.2	9.6
7	12.6	11.9	11.2
8	14.4	13.6	12.8
9	16.2	15.3	14.4

	9	**8**	**7**
1	.9	.8	.7
2	1.8	1.6	1.4
3	2.7	2.4	2.1
4	3.6	3.2	2.8
5	4.5	4.0	3.5
6	5.4	4.8	4.2
7	6.3	5.6	4.9
8	7.2	6.4	5.6
9	8.1	7.2	6.3

Prop. Parts

x	log sin	d	log cos	d	log tan	d	log cot	
30′	9.7844	17	9.8995	10	9.8850	26	0.1150	30′
40′	.7861	16	.8985	10	.8876	26	.1124	20′
50′	.7877	16	.8975	10	.8902	26	.1098	10′
38° 0′	9.7893	17	9.8965	10	9.8928	26	0.1072	**52° 0′**
10′	.7910	16	.8955	10	.8954	26	.1046	50′
20′	.7926	15	.8945	10	.8980	26	.1020	40′
30′	.7941	16	.8935	10	.9006	26	.0994	30′
40′	.7957	16	.8925	10	.9032	26	.0968	20′
50′	.7973	16	.8915	10	.9058	26	.0942	10′
39° 0′	9.7989	15	9.8905	10	9.9084	26	0.0916	**51° 0′**
10′	.8004	16	.8895	11	.9110	25	.0890	50′
20′	.8020	15	.8884	10	.9135	26	.0865	40′
30′	.8035	15	.8874	10	.9161	26	.0839	30′
40′	.8050	16	.8864	11	.9187	25	.0813	20′
50′	.8066	15	.8853	10	.9212	26	.0788	10′
40° 0′	9.8081	15	9.8843	11	9.9238	26	0.0762	**50° 0′**
10′	.8096	15	.8832	11	.9264	25	.0736	50′
20′	.8111	14	.8821	11	.9289	26	.0711	40′
30′	.8125	15	.8810	10	.9315	26	.0685	30′
40′	.8140	15	.8800	11	.9341	25	.0659	20′
50′	.8155	14	.8789	11	.9366	26	.0634	10′
41° 0′	9.8169	15	9.8778	11	9.9392	25	0.0608	**49° 0′**
10′	.8184	14	.8767	11	.9417	26	.0583	50′
20′	.8198	15	.8756	11	.9443	25	.0557	40′
30′	.8213	14	.8745	12	.9468	26	.0532	30′
40′	.8227	14	.8733	11	.9494	25	.0506	20′
50′	.8241	14	.8722	11	.9519	25	.0481	10′
42° 0′	9.8255	14	9.8711	12	9.9544	26	0.0456	**48° 0′**
10′	.8269	14	.8699	11	.9570	25	.0430	50′
20′	.8283	14	.8688	12	.9595	26	.0405	40′
30′	.8297	14	.8676	11	.9621	25	.0379	30′
40′	.8311	13	.8665	12	.9646	25	.0354	20′
50′	.8324	14	.8653	12	.9671	26	.0329	10′
43° 0′	9.8338	13	9.8641	12	9.9697	25	0.0303	**47° 0′**
10′	.8351	14	.8629	11	.9722	25	.0278	50′
20′	.8365	13	.8618	12	.9747	25	.0253	40′
30′	.8378	13	.8606	12	.9772	26	.0228	30′
40′	.8391	14	.8594	12	.9798	25	.0202	20′
50′	.8405	13	.8582	13	.9823	25	.0177	10′
44° 0′	9.8418	13	9.8569	12	9.9848	26	0.0152	**46° 0′**
10′	.8431	13	.8557	12	.9874	25	.0126	50′
20′	.8444	13	.8545	13	.9899	25	.0101	40′
30′	.8457	12	.8532	12	.9924	25	.0076	30′
40′	.8469	13	.8520	13	.9949	26	.0051	20′
50′	.8482	13	.8507	12	.9975	25	.0025	10′
45° 0′	9.8495		9.8495		0.0000		0.0000	**45° 0′**
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	17	16	15
1	1.7	1.6	1.5
2	3.4	3.2	3.0
3	5.1	4.8	4.5
4	6.8	6.4	6.0
5	8.5	8.0	7.5
6	10.2	9.6	9.0
7	11.9	11.2	10.5
8	13.6	12.8	12.0
9	15.3	14.4	13.5

	14	13	12
1	1.4	1.3	1.2
2	2.8	2.6	2.4
3	4.2	3.9	3.6
4	5.6	5.2	4.8
5	7.0	6.5	6.0
6	8.4	7.8	7.2
7	9.8	9.1	8.4
8	11.2	10.4	9.6
9	12.6	11.7	10.8

	11	10
1	1.1	1.0
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

30° – 45°
60° – 45°

Nat. tr funct'
0° – 1
90° – 7

15° – 30
75° – 60

30° – 4
60° – 4

Radia to degree and co versel Math const'.

Log. x, e^x, e^{-x}, n^2, n^3, $\sqrt{n}$, $\sqrt[3]{n}$

x	sin x	cos x	tan x	cot x	sec x	cosec x	
0° 0′	.00000	1.0000	.00000	∞	1.0000	∞	90° 0′
10′	.00291	1.0000	.00291	343.77	1.0000	343.78	50′
20′	.00582	1.0000	.00582	171.88	1.0000	171.89	40′
30′	.00873	1.0000	.00873	114.59	1.0000	114.59	30′
40′	.01164	.9999	.01164	85.940	1.0001	85.946	20′
50′	.01454	.9999	.01455	68.750	1.0001	68.757	10′
1° 0′	.01745	.9998	.01746	57.290	1.0002	57.299	89° 0′
10′	.02036	.9998	.02036	49.104	1.0002	49.114	50′
20′	.02327	.9997	.02328	42.964	1.0003	42.976	40′
30′	.02618	.9997	.02619	38.188	1.0003	38.202	30′
40′	.02908	.9996	.02910	34.368	1.0004	34.382	20′
50′	.03199	.9995	.03201	31.242	1.0005	31.258	10′
2° 0′	.03490	.9994	.03492	28.6363	1.0006	28.654	88° 0′
10′	.03781	.9993	.03783	26.4316	1.0007	26.451	50′
20′	.04071	.9992	.04075	24.5418	1.0008	24.562	40′
30′	.04362	.9990	.04366	22.9038	1.0010	22.926	30′
40′	.04653	.9989	.04658	21.4704	1.0011	21.494	20′
50′	.04943	.9988	.04949	20.2056	1.0012	20.230	10′
3° 0′	.05234	.9986	.05241	19.0811	1.0014	19.107	87° 0′
10′	.05524	.9985	.05533	18.0750	1.0015	18.103	50′
20′	.05814	.9983	.05824	17.1693	1.0017	17.198	40′
30′	.06105	.9981	.06116	16.3499	1.0019	16.380	30′
40′	.06395	.9980	.06408	15.6048	1.0021	15.637	20′
50′	.06685	.9978	.06700	14.9244	1.0022	14.958	10′
4° 0′	.06976	.9976	.06993	14.3007	1.0024	14.336	86° 0′
10′	.07266	.9974	.07285	13.7267	1.0027	13.763	50′
20′	.07556	.9971	.07578	13.1969	1.0029	13.235	40′
30′	.07846	.9969	.07870	12.7062	1.0031	12.746	30′
40′	.08136	.9967	.08163	12.2505	1.0033	12.291	20′
50′	.08426	.9964	.08456	11.8262	1.0036	11.868	10′
5° 0′	.08716	.9962	.08749	11.4301	1.0038	11.474	85° 0′
10′	.09005	.9959	.09042	11.0594	1.0041	11.105	50′
20′	.09295	.9957	.09335	10.7119	1.0044	10.758	40′
30′	.09585	.9954	.09629	10.3854	1.0046	10.433	30′
40′	.09874	.9951	.09923	10.0780	1.0049	10.128	20′
50′	.10164	.9948	.10216	9.7882	1.0052	9.839	10′
6° 0′	.10453	.9945	.10510	9.5144	1.0055	9.5668	84° 0′
10′	.10742	.9942	.10805	9.2553	1.0058	9.3092	50′
20′	.11031	.9939	.11099	9.0098	1.0061	9.0652	40′
30′	.11320	.9936	.11394	8.7769	1.0065	8.8337	30′
40′	.11609	.9932	.11688	8.5555	1.0068	8.6138	20′
50′	.11898	.9929	.11983	8.3450	1.0072	8.4647	10′
7° 0′	.12187	.9925	.12278	8.1443	1.0075	8.2055	83° 0′
10′	.12476	.9922	.12574	7.9530	1.0079	8.0157	50′
20′	.12764	.9918	.12869	7.7704	1.0083	7.8344	40′
30′	.13053	.9914	.13165	7.5958	1.0086	7.6613	30′
	cos X	sin X	cot X	tan X	cosec X	sec X	X

x	sin x	cos x	tan x	cot x	sec x	cosec x	
30′	.1305	.9914	.1317	7.5958	1.0086	7.6613	30′
40′	.1334	.9911	.1346	7.4287	1.0090	7.4957	20′
50′	.1363	.9907	.1376	7.2687	1.0094	7.3372	10′
8° 0′	.1392	.9903	.1405	7.1154	1.0098	7.1853	**82° 0′**
10′	.1421	.9899	.1435	6.9682	1.0102	7.0396	50′
20′	.1449	.9894	.1465	6.8269	1.0107	6.8998	40′
30′	.1478	.9890	.1495	6.6912	1.0111	6.7655	30′
40′	.1507	.9886	.1524	6.5606	1.0116	6.6363	20′
50′	.1536	.9881	.1554	6.4348	1.0120	6.5121	10′
9° 0′	.1564	.9877	.1584	6.3138	1.0125	6.3925	**81° 0′**
10′	.1593	.9872	.1614	6.1970	1.0129	6.2772	50′
20′	.1622	.9868	.1644	6.0844	1.0134	6.1661	40′
30′	.1650	.9863	.1673	5.9758	1.0139	6.0589	30′
40′	.1679	.9858	.1703	5.8708	1.0144	5.9554	20′
50′	.1708	.9853	.1733	5.7694	1.0149	5.8554	10′
10° 0′	.1736	.9848	.1763	5.6713	1.0154	5.7588	**80° 0′**
10′	.1765	.9843	.1793	5.5764	1.0160	5.6653	50′
20′	.1794	.9838	.1823	5.4845	1.0165	5.5749	40′
30′	.1822	.9833	.1853	5.3955	1.0170	5.4874	30′
40′	.1851	.9827	.1883	5.3093	1.0176	5.4026	20′
50′	.1880	.9822	.1914	5.2257	1.0182	5.3205	10′
11° 0′	.1908	.9816	.1944	5.1446	1.0187	5.2408	**79° 0′**
10′	.1937	.9811	.1974	5.0658	1.0193	5.1636	50′
20′	.1965	.9805	.2004	4.9894	1.0199	5.0886	40′
30′	.1994	.9799	.2035	4.9152	1.0205	5.0159	30′
40′	.2022	.9793	.2065	4.8430	1.0211	4.9452	20′
50′	.2051	.9787	.2095	4.7729	1.0217	4.8765	10′
12° 0′	.2079	.9781	.2126	4.7046	1.0223	4.8097	**78° 0′**
10′	.2108	.9775	.2156	4.6382	1.0230	4.7448	50′
20′	.2136	.9769	.2186	4.5736	1.0236	4.6817	40′
30′	.2164	.9763	.2217	4.5107	1.0243	4.6202	30′
40′	.2193	.9757	.2247	4.4494	1.0249	4.5604	20′
50′	.2221	.9750	.2278	4.3897	1.0256	4.5022	10′
13° 0′	.2250	.9744	.2309	4.3315	1.0263	4.4454	**77° 0′**
10′	.2278	.9737	.2339	4.2747	1.0270	4.3901	50′
20′	.2306	.9730	.2370	4.2193	1.0277	4.3362	40′
30′	.2334	.9724	.2401	4.1653	1.0284	4.2837	30′
40′	.2363	.9717	.2432	4.1126	1.0291	4.2324	20′
50′	.2391	.9710	.2462	4.0611	1.0299	4.1824	10′
14° 0′	.2419	.9703	.2493	4.0108	1.0306	4.1336	**76° 0′**
10′	.2447	.9696	.2524	3.9617	1.0314	4.0859	50′
20′	.2476	.9689	.2555	3.9136	1.0321	4.0394	40′
30′	.2504	.9681	.2586	3.8667	1.0329	3.9939	30′
40′	.2532	.9674	.2617	3.8208	1.0337	3.9495	20′
50′	.2560	.9667	.2648	3.7760	1.0345	3.9061	10′
15° 0′	.2588	.9659	.2679	3.7321	1.0353	3.8637	**75° 0′**
	cos X	sin X	cot X	tan X	cosec X	sec X	X

x		sin x	cos x	tan x	cot x	sec x	cosec x		
15°	**0′**	.2588	.9659	.2679	3.7321	1.0353	3.8637	**75°**	**0′**
	10′	.2616	.9652	.2711	3.6891	1.0361	3.8222		50′
	20′	.2644	.9644	.2742	3.6470	1.0369	3.7817		40′
	30′	.2672	.9636	.2773	3.6059	1.0377	3.7420		30′
	40′	.2700	.9628	.2805	3.5656	1.0386	3.7032		20′
	50′	.2728	.9621	.2836	3.5261	1.0394	3.6652		10′
16°	**0′**	.2756	.9613	.2867	3.4874	1.0403	3.6280	**74°**	**0′**
	10′	.2784	.9605	.2899	3.4495	1.0412	3.5915		50′
	20′	.2812	.9596	.2931	3.4124	1.0421	3.5559		40′
	30′	.2840	.9588	.2962	3.3759	1.0430	3.5209		30′
	40′	.2868	.9580	.2994	3.3402	1.0439	3.4867		20′
	50′	.2896	.9572	.3026	3.3052	1.0448	3.4532		10′
17°	**0′**	.2924	.9563	.3057	3.2709	1.0457	3.4203	**73°**	**0′**
	10′	.2952	.9555	.3089	3.2371	1.0466	3.3881		50′
	20′	.2979	.9546	.3121	3.2041	1.0476	3.3565		40′
	30′	.3007	.9537	.3153	3.1716	1.0485	3.3255		30′
	40′	.3035	.9528	.3185	3.1397	1.0495	3.2951		20′
	50′	.3062	.9520	.3217	3.1084	1.0505	3.2653		10′
18°	**0′**	.3090	.9511	.3249	3.0777	1.0515	3.2361	**72°**	**0′**
	10′	.3118	.9502	.3281	3.0475	1.0525	3.2074		50′
	20′	.3145	.9492	.3314	3.0178	1.0535	3.1792		40′
	30′	.3173	.9483	.3346	2.9887	1.0545	3.1516		30′
	40′	.3201	.9474	.3378	2.9600	1.0555	3.1244		20′
	50′	.3228	.9465	.3411	2.9319	1.0566	3.0977		10′
19°	**0′**	.3256	.9455	.3443	2.9042	1.0576	3.0716	**71°**	**0′**
	10′	.3283	.9446	.3476	2.8770	1.0587	3.0458		50′
	20′	.3311	.9436	.3508	2.8502	1.0598	3.0206		40′
	30′	.3338	.9426	.3541	2.8239	1.0609	2.9957		30′
	40′	.3365	.9417	.3574	2.7980	1.0620	2.9714		20′
	50′	.3393	.9407	.3607	2.7725	1.0631	2.9474		10′
20°	**0′**	.3420	.9397	.3640	2.7475	1.0642	2.9238	**70°**	**0′**
	10′	.3448	.9387	.3673	2.7228	1.0653	2.9006		50′
	20′	.3475	.9377	.3706	2.6985	1.0665	2.8779		40′
	30′	.3502	.9367	.3739	2.6746	1.0676	2.8555		30′
	40′	.3529	.9356	.3772	2.6511	1.0688	2.8334		20′
	50′	.3557	.9346	.3805	2.6279	1.0700	2.8118		10′
21°	**0′**	.3584	.9336	.3839	2.6051	1.0712	2.7904	**69°**	**0′**
	10′	.3611	.9325	.3872	2.5826	1.0724	2.7695		50′
	20′	.3638	.9315	.3906	2.5605	1.0736	2.7488		40′
	30′	.3665	.9304	.3939	2.5386	1.0748	2.7285		30′
	40′	.3692	.9293	.3973	2.5172	1.0760	2.7085		20′
	50′	.3719	.9283	.4006	2.4960	1.0773	2.6888		10′
22°	**0′**	.3746	.9272	.4040	2.4751	1.0785	2.6695	**68°**	**0′**
	10′	.3773	.9261	.4074	2.4545	1.0798	2.6504		50′
	20′	.3800	.9250	.4108	2.4342	1.0811	2.6316		40′
	30′	.3827	.9239	.4142	2.4142	1.0824	2.6131		30′
		cos X	sin X	cot X	tan X	cosec X	sec X	X	

x	sin *x*	cos *x*	tan *x*	cot *x*	sec *x*	cosec *x*	
30′	.3827	.9239	.4142	2.4142	1.0824	2.6131	30′
40′	.3854	.9228	.4176	2.3945	1.0837	2.5949	20′
50′	.3881	.9216	.4210	2.3750	1.0850	2.5770	10′
23° 0′	.3907	.9205	.4245	2.3559	1.0864	2.5593	**67°** 0′
10′	.3934	.9194	.4279	2.3369	1.0877	2.5419	50′
20′	.3961	.9182	.4314	2.3183	1.0891	2.5247	40′
30′	.3987	.9171	.4348	2.2998	1.0904	2.5078	30′
40′	.4014	.9159	.4383	2.2817	1.0918	2.4912	20′
50′	.4041	.9147	.4417	2.2637	1.0932	2.4748	10′
24° 0′	.4067	.9135	.4452	2.2460	1.0946	2.4586	**66°** 0′
10′	.4094	.9124	.4487	2.2286	1.0961	2.4426	50′
20′	.4120	.9112	.4522	2.2113	1.0975	2.4269	40′
30′	.4147	.9100	.4557	2.1943	1.0990	2.4114	30′
40′	.4173	.9088	.4592	2.1775	1.1004	2.3961	20′
50′	.4200	.9075	.4628	2.1609	1.1019	2.3811	10′
25° 0′	.4226	.9063	.4663	2.1445	1.1034	2.3662	**65°** 0′
10′	.4253	.9051	.4699	2.1283	1.1049	2.3515	50′
20′	.4279	.9038	.4734	2.1123	1.1064	2.3371	40′
30′	.4305	.9026	.4770	2.0965	1.1079	2.3228	30′
40′	.4331	.9013	.4806	2.0809	1.1095	2.3088	20′
50′	.4358	.9001	.4841	2.0655	1.1110	2.2949	10′
26° 0′	.4384	.8988	.4877	2.0503	1.1126	2.2812	**64°** 0′
10′	.4410	.8975	.4913	2.0353	1.1142	2.2677	50′
20′	.4436	.8962	.4950	2.0204	1.1158	2.2543	40′
30′	.4462	.8949	.4986	2.0057	1.1174	2.2412	30′
40′	.4488	.8936	.5022	1.9912	1.1190	2.2282	20′
50′	.4514	.8923	.5059	1.9768	1.1207	2.2154	10′
27° 0′	.4540	.8910	.5095	1.9626	1.1223	2.2027	**63°** 0′
10′	.4566	.8897	.5132	1.9486	1.1240	2.1902	50′
20′	.4592	.8884	.5169	1.9347	1.1257	2.1779	40′
30′	.4617	.8870	.5206	1.9210	1.1274	2.1657	30′
40′	.4643	.8857	.5243	1.9074	1.1291	2.1537	20′
50′	.4669	.8843	.5280	1.8940	1.1308	2.1418	10′
28° 0′	.4695	.8829	.5317	1.8807	1.1326	2.1301	**62°** 0′
10′	.4720	.8816	.5354	1.8676	1.1343	2.1185	50′
20′	.4746	.8802	.5392	1.8546	1.1361	2.1070	40′
30′	.4772	.8788	.5430	1.8418	1.1379	2.0957	30′
40′	.4797	.8774	.5467	1.8291	1.1397	2.0846	20′
50′	.4823	.8760	.5505	1.8165	1.1415	2.0736	10′
29° 0′	.4848	.8746	.5543	1.8040	1.1434	2.0627	**61°** 0′
10′	.4874	.8732	.5581	1.7917	1.1452	2.0519	50′
20′	.4899	.8718	.5619	1.7796	1.1471	2.0413	40′
30′	.4924	.8704	.5658	1.7675	1.1490	2.0308	30′
40′	.4950	.8689	.5696	1.7556	1.1509	2.0204	20′
50′	.4975	.8675	.5735	1.7437	1.1528	2.0101	10′
30° 0′	.5000	.8660	.5774	1.7321	1.1547	2.0000	**60°** 0′
	cos *X*	sin *X*	cot *X*	tan *X*	cosec *X*	sec *X*	*X*

15° — 3(
75° — 6(
30° — 4
60° — 4
Radia to degre and co. versel Math const'
Log_e x, e^x, e^-x, n², n³, √n, ∛n

x		sin x	cos x	tan x	cot x	sec x	cosec x		
30°	0′	.5000	.8660	.5774	1.7321	1.1547	2.0000	60°	0′
	10′	.5025	.8646	.5812	1.7205	1.1567	1.9900		50′
	20′	.5050	.8631	.5851	1.7090	1.1586	1.9801		40′
	30′	.5075	.8616	.5890	1.6977	1.1606	1.9703		30′
	40′	.5100	.8601	.5930	1.6864	1.1626	1.9606		20′
	50′	.5125	.8587	.5969	1.6753	1.1646	1.9511		10′
31°	0′	.5150	.8572	.6009	1.6643	1.1666	1.9416	59°	0′
	10′	.5175	.8557	.6048	1.6534	1.1687	1.9323		50′
	20′	.5200	.8542	.6088	1.6426	1.1708	1.9230		40′
	30′	.5225	.8526	.6128	1.6319	1.1728	1.9139		30′
	40′	.5250	.8511	.6168	1.6212	1.1749	1.9049		20′
	50′	.5275	.8496	.6208	1.6107	1.1770	1.8959		10′
32°	0′	.5299	.8480	.6249	1.6003	1.1792	1.8871	58°	0′
	10′	.5324	.8465	.6289	1.5900	1.1813	1.8783		50′
	20′	.5348	.8450	.6330	1.5798	1.1835	1.8699		40′
	30′	.5373	.8434	.6371	1.5697	1.1857	1.8612		30′
	40′	.5398	.8418	.6412	1.5597	1.1879	1.8527		20′
	50′	.5422	.8403	.6453	1.5497	1.1901	1.8444		10′
33°	0′	.5446	.8387	.6494	1.5399	1.1924	1.8361	57°	0′
	10′	.5471	.8371	.6536	1.5301	1.1946	1.8279		50′
	20′	.5495	.8355	.6577	1.5204	1.1969	1.8198		40′
	30′	.5519	.8339	.6619	1.5108	1.1992	1.8118		30′
	40′	.5544	.8323	.6661	1.5013	1.2015	1.8039		20′
	50′	.5568	.8307	.6703	1.4919	1.2039	1.7960		10′
34°	0′	.5592	.8290	.6745	1.4826	1.2062	1.7883	56°	0′
	10′	.5616	.8274	.6787	1.4733	1.2086	1.7806		50′
	20′	.5640	.8258	.6830	1.4641	1.2110	1.7730		40′
	30′	.5664	.8241	.6873	1.4550	1.2134	1.7655		30′
	40′	.5688	.8225	.6916	1.4460	1.2158	1.7581		20′
	50′	.5712	.8208	.6959	1.4370	1.2183	1.7507		10′
35°	0′	.5736	.8192	.7002	1.4281	1.2208	1.7435	55°	0′
	10′	.5760	.8175	.7046	1.4193	1.2233	1.7362		50′
	20′	.5783	.8158	.7089	1.4106	1.2258	1.7291		40′
	30′	.5807	.8141	.7133	1.4019	1.2283	1.7221		30′
	40′	.5831	.8124	.7177	1.3934	1.2309	1.7151		20′
	50′	.5854	.8107	.7221	1.3848	1.2335	1.7082		10′
36°	0′	.5878	.8090	.7265	1.3764	1.2361	1.7013	54°	0′
	10′	.5901	.8073	.7310	1.3680	1.2387	1.6945		50′
	20′	.5925	.8056	.7355	1.3597	1.2413	1.6878		40′
	30′	.5948	.8039	.7400	1.3514	1.2440	1.6812		30′
	40′	.5972	.8021	.7445	1.3432	1.2467	1.6746		20′
	50′	.5995	.8004	.7490	1.3351	1.2494	1.6681		10′
37°	0′	.6018	.7986	.7536	1.3270	1.2521	1.6616	53°	0′
	10′	.6041	.7969	.7581	1.3190	1.2549	1.6553		50′
	20′	.6065	.7951	.7627	1.3111	1.2577	1.6489		40′
	30′	.6088	.7934	.7673	1.3032	1.2605	1.6427		30′
		cos X	sin X	cot X	tan X	cosec X	sec X	X	

x		sin x	cos x	tan x	cot x	sec x	cosec x		
	30′	.6088	.7934	.7673	1.3032	1.2605	1.6427		30′
	40′	.6111	.7916	.7720	1.2954	1.2633	1.6365		20′
	50′	.6134	.7898	.7766	1.2876	1.2662	1.6304		10′
38°	**0′**	.6157	.7880	.7813	1.2799	1.2690	1.6243	**52°**	**0′**
	10′	.6180	.7862	.7860	1.2723	1.2719	1.6183		50′
	20′	.6202	.7844	.7907	1.2647	1.2748	1.6123		40′
	30′	.6225	.7826	.7954	1.2572	1.2779	1.6064		30′
	40′	.6248	.7808	.8002	1.2497	1.2808	1.6005		20′
	50′	.6271	.7790	.8050	1.2423	1.2837	1.5948		10′
39°	**0′**	.6293	.7771	.8098	1.2349	1.2868	1.5890	**51°**	**0′**
	10′	.6316	.7753	.8146	1.2276	1.2898	1.5833		50′
	20′	.6338	.7735	.8195	1.2203	1.2929	1.5777		40′
	30′	.6361	.7716	.8243	1.2131	1.2960	1.5721		30′
	40′	.6383	.7698	.8292	1.2059	1.2991	1.5666		20′
	50′	.6406	.7679	.8342	1.1988	1.3022	1.5611		10′
40°	**0′**	.6428	.7660	.8391	1.1918	1.3054	1.5557	**50°**	**0′**
	10′	.6450	.7642	.8441	1.1847	1.3086	1.5504		50′
	20′	.6472	.7623	.8491	1.1778	1.3118	1.5450		40′
	30′	.6494	.7604	.8541	1.1708	1.3151	1.5398		30′
	40′	.6517	.7585	.8591	1.1640	1.3184	1.5346		20′
	50′	.6539	.7566	.8642	1.1571	1.3217	1.5294		10′
41°	**0′**	.6561	.7547	.8693	1.1504	1.3250	1.5243	**49°**	**0′**
	10′	.6583	.7528	.8744	1.1436	1.3284	1.5192		50′
	20′	.6604	.7509	.8796	1.1369	1.3318	1.5142		40′
	30′	.6626	.7490	.8847	1.1303	1.3352	1.5092		30′
	40′	.6648	.7470	.8899	1.1237	1.3386	1.5042		20′
	50′	.6670	.7451	.8952	1.1171	1.3421	1.4993		10′
42°	**0′**	.6691	.7431	.9004	1.1106	1.3456	1.4945	**48°**	**0′**
	10′	.6713	.7412	.9057	1.1041	1.3492	1.4897		50′
	20′	.6734	.7392	.9110	1.0977	1.3527	1.4849		40′
	30′	.6756	.7373	.9163	1.0913	1.3563	1.4802		30′
	40′	.6777	.7353	.9217	1.0850	1.3600	1.4755		20′
	50′	.6799	.7333	.9271	1.0786	1.3636	1.4709		10′
43°	**0′**	.6820	.7314	.9325	1.0724	1.3673	1.4663	**47°**	**0′**
	10′	.6841	.7294	.9380	1.0661	1.3711	1.4617		50′
	20′	.6862	.7274	.9435	1.0599	1.3748	1.4572		40′
	30′	.6884	.7254	.9490	1.0538	1.3786	1.4527		30′
	40′	.6905	.7234	.9545	1.0477	1.3824	1.4483		20′
	50′	.6926	.7214	.9601	1.0416	1.3863	1.4439		10′
44°	**0′**	.6947	.7193	.9657	1.0355	1.3902	1.4396	**46°**	**0′**
	10′	.6967	.7173	.9713	1.0295	1.3941	1.4352		50′
	20′	.6988	.7153	.9770	1.0235	1.3980	1.4310		40′
	30′	.7009	.7133	.9827	1.0176	1.4020	1.4267		30′
	40′	.7030	.7112	.9884	1.0117	1.4061	1.4225		20′
	50′	.7050	.7092	.9942	1.0058	1.4101	1.4184		10′
45°	**0′**	.7071	.7071	1.0000	1.0000	1.4142	1.4142	**45°**	**0′**
		cos X	**sin X**	**cot X**	**tan X**	**cosec X**	**sec X**	X	

n	n degrees into radians	n minutes into radians	n seconds into radians	n	n radians into degree measure
0	0.00000	0.00000	0.00000		
1	0.01745	0.00029	0.00000	0.00001	0° 0′ 02″
2	0.03491	0.00058	0.00001	0.00002	0 0 04
3	0.05236	0.00087	0.00001	0.00003	0 0 06
4	0.06981	0.00116	0.00002	0.00004	0 0 08
5	0.08727	0.00145	0.00002	0.00005	0° 0′ 10″
6	0.10472	0.00175	0.00003	0.00006	0 0 12
7	0.12217	0.00204	0.00003	0.00007	0 0 14
8	0.13963	0.00233	0.00004	0.00008	0 0 17
9	0.15708	0.00262	0.00004	0.00009	0 0 19
10	0.17453	0.00291	0.00005		
11	0.19199	0.00320	0.00005	0.0001	0° 0′ 21″
12	0.20944	0.00349	0.00006	0.0002	0 0 41″
13	0.22689	0.00378	0.00006	0.0003	0 1 02
14	0.24435	0.00407	0.00007	0.0004	0 1 23
15	0.26180	0.00436	0.00007	0.0005	0° 1′ 43″
16	0.27925	0.00465	0.00008	0.0006	0 2 04
17	0.29671	0.00495	0.00008	0.0007	0 2 24
18	0.31416	0.00524	0.00009	0.0008	0 2 45
19	0.33161	0.00553	0.00009	0.0009	0 3 06
20	0.34907	0.00582	0.00010		
21	0.36652	0.00611	0.00010	0.001	0° 03′ 26″
22	0.38397	0.00640	0.00011	0.002	0 06 53
23	0.40143	0.00669	0.00011	0.003	0 10 19
24	0.41888	0.00698	0.00012	0.004	0 13 45
25	0.43633	0.00727	0.00012	0.005	0° 17′ 11″
26	0.45379	0.00756	0.00013	0.006	0 20 38
27	0.47124	0.00785	0.00013	0.007	0 24 04
28	0.48869	0.00814	0.00014	0.008	0 27 30
29	0.50615	0.00844	0.00014	0.009	0 30 56
30	0.52360	0.00873	0.00015		
31	0.54105	0.00902	0.00015	0.01	0° 34′ 23″
32	0.55851	0.00931	0.00016	0.02	1 08 45
33	0.57596	0.00960	0.00016	0.03	1 43 08
34	0.59341	0.00989	0.00016	0.04	2 17 31
35	0.61087	0.01018	0.00017	0.05	2° 51′ 53″
36	0.62832	0.01047	0.00017	0.06	3 26 16
37	0.64577	0.01076	0.00018	0.07	4 00 39
38	0.66323	0.01105	0.00018	0.08	4 35 01
39	0.68068	0.01134	0.00019	0.09	5 09 24
40	0.69813	0.01164	0.00019		
41	0.71558	0.01193	0.00020	0.1	5° 43 46″
42	0.73304	0.01222	0.00020	0.2	11 27 33
43	0.75049	0.01251	0.00021	0.3	17 11 19
44	0.76794	0.01280	0.00021	0.4	22 55 6

n	n degrees into radians	n minutes into radians	n seconds into radians	n	n radians into degree measure
45	0.78540	0.01309	0.00022	0.5	28° 38′ 52″
46	0.80285	0.01338	0.00022	0.6	34 22 39
47	0.82030	0.01367	0.00023	0.7	40 06 25
48	0.83776	0.01396	0.00023	0.8	45 50 12
49	0.85521	0.01425	0.00024	0.9	51 33 58
50	0.87266	0.01454	0.00024		
51	0.89012	0.01484	0.00025	1.0	57° 17′ 45″
52	0.90757	0.01513	0.00025	2.0	114 35 30
53	0.92502	0.01542	0.00026	3.0	171 53 14
54	0.94248	0.01571	0.00026	4.0	229 10 59
55	0.95993	0.01600	0.00027	5.0	286° 28′ 44″
56	0.97738	0.01629	0.00027	6.0	343 46 29
57	0.99484	0.01658	0.00028	7.0	401 04 14
58	1.01229	0.01687	0.00028	8.0	458 21 58
59	1.02974	0.01716	0.00029	9.0	515 39 43
60	1.04720	0.01745	0.00029	10.0	572° 57′ 28″

TABLE V. MATHEMATICAL CONSTANTS

$\pi = 3.14159\ 26535\ 89793.$ $\quad \frac{1}{\pi} = 0.31830\ 98861\ 83791.$

$\pi^2 = 9.86960\ 44010\ 89359.$ $\quad \frac{1}{\pi^2} = 0.10132\ 11836\ 42338.$

$\pi^3 = 31.00627\ 66802\ 99820.$ $\quad \frac{1}{\pi^3} = 0.03225\ 15344\ 33199.$

$\sqrt{\pi} = 1.77245\ 38509\ 05516.$ $\quad \frac{1}{\sqrt{\pi}} = 0.56418\ 95835\ 47756.$

$$1 \text{ radian} = \frac{180^\circ}{\pi} = 57^\circ.29577\ 95131,$$

$$= \frac{10800'}{\pi} = 3437'.74677\ 07849,$$

$$= \frac{648000''}{\pi} = 206264''.80624\ 70964.$$

radians.	radians.
$1^\circ = 0.01745\ 32925\ 19943.$	$1' = 0.00029\ 08882\ 08666.$
$(1^\circ)^2 = 0.00030\ 46174\ 19787.$	$(1')^2 = 0.00000\ 00846\ 15950$
$(1^\circ)^3 = 0.00000\ 53165\ 76934.$	$(1')^3 = 0.00000\ 00000\ 24614.$

$1'' = 0.00000\ 48481\ 36811$

$(1'')^2 = 0.00000\ 00000\ 23504.$

$\sin 1^\circ = 0.01745\ 24064\ 37284.$

$\sin 1' = 0.00029\ 08882\ 04563.$

$\sin 1'' = 0.00000\ 48481\ 36811.$

$$e = \text{Naperian base} = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots = 2.71828\ 18284\ 59045.$$

$M = 0.43429\ 44819\ 03252;\ \log_{10} n = M \log_e n.$

$\frac{1}{M} = 2.30258\ 50929\ 94046;\ \log_e n = \frac{1}{M} \log_{10} n.$

Radia[ns] to degre[es] and co[n]versel[y]. Math const's.
$\log_e x$, e^x, e^{-x}, n^2, n^3, $\sqrt{n}$, $\sqrt[3]{n}$

x	$\log_e x$	e^x	e^{-x}	x	$\log_e x$	e^x	e^{-x}
0.00	$-\infty$	1.000	1.000	2.50	0.916	12.18	0.082
0.05	−2.996	1.051	0.951	2.55	0.936	12.81	0.078
0.10	−2.303	1.105	0.905	2.60	0.956	13.46	0.074
0.15	−1.897	1.162	0.861	2.65	0.975	14.15	0.071
0.20	−1.610	1.221	0.819	2.70	0.993	14.88	0.067
0.25	−1.386	1.284	0.779	2.75	1.012	15.64	0 064
0.30	−1.204	1.350	0.741	2.80	1.030	16.44	0.061
0.35	−1.050	1.419	0.705	2.85	1.047	17.29	0.058
0.40	−0.916	1.492	0.670	2.90	1.065	18.17	0.055
0.45	−0.799	1.568	0.638	2.95	1.082	19.11	0.052
0.50	−0.693	1.649	0.607	3.00	1.099	20.09	0.050
0.55	−0.598	1.733	0.577	3.05	1.115	21.12	0.047
0.60	−0.511	1.822	0.549	3.10	1.131	22.20	0.045
0.65	−0.431	1.916	0.522	3.15	1.147	23.34	0.043
0.70	−0.357	2.014	0.497	3.20	1.163	24.53	0.041
0.75	−0.288	2.117	0.472	3.25	1.179	25.79	0.039
0.80	−0.223	2.226	0.449	3.30	1.194	27.11	0.037
0.85	−0.163	2.340	0.427	3.35	1.209	28.50	0.035
0.90	−0.105	2.460	0.407	3.40	1.224	29.96	0.033
0.95	−0.051	2.586	0.387	3.45	1.238	31.50	0.032
1.00	0.000	2.718	0.368	3.50	1.253	33.12	0.030
1.05	+0.049	2.858	0.350	3.55	1.267	34.81	0.029
1.10	0.095	3.004	0.333	3.60	1.281	36.60	0.027
1.15	0.140	3.158	0.317	3.65	1.295	38.47	0.026
1.20	0.182	3.320	0.301	3.70	1.308	40.45	0.025
1.25	0.223	3.490	0.287	3.75	1.322	42.52	0.024
1.30	0.262	3.669	0.273	3.80	1.335	44.70	0.022
1.35	0.300	3.857	0.259	3.85	1.348	46.99	0.021
1.40	0.337	4.055	0.247	3.90	1.361	49.40	0.020
1.45	0.372	4.263	0.235	3.95	1.374	51.94	0.019
1.50	0.406	4.482	0.223	4.00	1.386	54.60	0.018
1.55	0.438	4.711	0.212	4.05	1.399	57.40	0.017
1.60	0.470	4.953	0.202	4.10	1.411	60.34	0.017
1.65	0.501	5.207	0.192	4.15	1.423	63.43	0.016
1.70	0.531	5.474	0.183	4.20	1.435	66.69	0.015
1.75	0.560	5.755	0.174	4.25	1.447	70.11	0.014
1.80	0.588	6.050	0.165	4.30	1.459	73.70	0.014
1.85	0.615	6.360	0.157	4.35	1.470	77.48	0.013
1.90	0.642	6.686	0.150	4.40	1.482	81.45	0.012
1.95	0.668	7.029	0.142	4.45	1.493	85.63	0.012
2.00	0.693	7.389	0.135	4.50	1.504	90.02	0.011
2.05	0.718	7.768	0.129	4.55	1.515	94.63	0.011
2.10	0.742	8.166	0.122	4.60	1.526	99.48	0.010
2.15	0.766	8.585	0.116	4.65	1.537	104.58	0.010
2.20	0.789	9.025	0.111	4.70	1.548	109.95	0.009
2.25	0.811	9.488	0.105	4.75	1.558	115.58	0 009
2.30	0.833	9.974	0.100	4.80	1.569	121.51	0.008
2.35	0.854	10.486	0.095	4.85	1.579	127.74	0.008
2.40	0.876	11.023	0.091	4.90	1.589	134.29	0.007
2.45	0.896	11.588	0.086	4.95	1.599	141.17	0.007
2.50	0.916	12.182	0.082	5.00	1.609	148.41	0.007

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$
1	1	1	1	1	**51**	2601	132651	7.141	3.708
2	4	8	1.414	1.260	52	2704	140608	7.211	3.733
3	9	27	1.732	1.442	53	2809	148877	7.280	3.756
4	16	64	2.000	1.587	54	2916	157464	7.348	3.780
5	25	125	2.236	1.710	55	3025	166375	7.416	3.803
6	36	216	2.449	1.817	56	3136	175616	7.483	3.826
7	49	343	2.646	1.913	57	3249	185193	7.550	3.849
8	64	512	2.828	2.000	58	3364	195112	7.616	3.871
9	81	729	3.000	2.080	59	3481	205379	7.681	3.893
10	100	1000	3.162	2.154	60	3600	216000	7.746	3.915
11	121	1331	3.317	2.224	**61**	3721	226981	7.810	3.936
12	144	1728	3.464	2.289	62	3844	238328	7.874	3.958
13	169	2197	3.606	2.351	63	3969	250047	7.937	3.979
14	196	2744	3.742	2.410	64	4096	262144	8.000	4.000
15	225	3375	3.873	2.466	65	4225	274625	8.062	4.021
16	256	4096	4.000	2.520	66	4356	287496	8.124	4.041
17	289	4913	4.123	2.571	67	4489	300763	8.185	4.062
18	324	5832	4.243	2.621	68	4624	314432	8.246	4.082
19	361	6859	4.359	2.668	69	4761	328509	8.307	4.102
20	400	8000	4.472	2.714	70	4900	343000	8.367	4.121
21	441	9261	4.583	2.759	**71**	5041	357911	8.426	4.141
22	484	10648	4.690	2.802	72	5184	373248	8.485	4.160
23	529	12167	4.796	2.844	73	5329	389017	8.544	4.179
24	576	13824	4.899	2.884	74	5476	405224	8.602	4.198
25	625	15625	5.000	2.924	75	5625	421875	8.660	4.217
26	676	17576	5.099	2.962	76	5776	438976	8.718	4.236
27	729	19683	5.196	3.000	77	5929	456533	8.775	4.254
28	784	21952	5.291	3.037	78	6084	474552	8.832	4.273
29	841	24389	5.385	3.072	79	6241	493039	8.888	4.291
30	900	27000	5.477	3.107	80	6400	512000	8.944	4.309
31	961	29791	5.568	3.141	**81**	6561	531441	9.000	4.327
32	1024	32768	5.657	3.175	82	6724	551368	9.055	4.344
33	1089	35937	5.745	3.208	83	6889	571787	9.110	4.362
34	1156	39304	5.831	3.240	84	7056	592704	9.165	4.380
35	1225	42875	5.916	3.271	85	7225	614125	9.220	4.397
36	1296	46656	6.000	3.302	86	7396	636056	9.274	4.414
37	1369	50653	6.083	3.332	87	7569	658503	9.327	4.431
38	1444	54872	6.164	3.362	88	7744	681472	9.381	4.448
39	1521	59319	6.245	3.391	89	7921	704969	9.434	4.465
40	1600	64000	6.325	3.420	90	8100	729000	9.487	4.481
41	1681	68921	6.403	3.448	**91**	8281	753571	9.539	4.498
42	1764	74088	6.481	3.476	92	8464	778688	9.592	4.514
43	1849	79507	6.557	3.503	93	8649	804357	9.644	4.531
44	1936	85184	6.633	3.530	94	8836	830584	9.695	4.547
45	2025	91125	6.708	3.557	95	9025	857375	9.747	4.563
46	2116	97336	6.782	3.583	96	9216	884736	9.798	4.579
47	2209	103823	6.856	3.609	97	9409	912673	9.849	4.595
48	2304	110592	6.928	3.634	98	9604	941192	9.899	4.610
49	2401	117649	7.000	3.659	99	9801	970299	9.950	4.626
50	2500	125000	7.071	3.684	100	10000	1000000	10.000	4.642
n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$